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COVARIANT ANALYSIS OF NEWTONIAN MULTI-FLUID MODELS FOR NEUTRON STARS: II STRESS–ENERGY TENSORS AND VIRIAL THEOREMS

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The 4-dimensionally covariant approach to multiconstituent Newtonian fluid dynamics presented in the preceding paper of this series is developed by construction of the relevant 4-dimensional stress–energy tensor whose conservation in the non-dissipative variational case is shown to be interpretable as a Noether identity of the Milne spacetime structure. The formalism is illustrated by the application to homogeneously expanding cosmological models, for which appropriately generalized local Bernoulli constants are constructed. Another application is to the Iordanski type generalization of the Joukowski formula for the Magnus force on a vortex. Finally, at a global level, a new (formally simpler but more generally applicable) version of the "virial theorem" is obtained for multiconstituent — neutron or other — fluid star models as a special case within an extensive category of formulae whereby the time evolution of variously weighted mass moment integrals is determined by corresponding space integrals of stress tensor components, with the implication that all such stress integrals must vanish for any stationary equilibrium configuration.

Keywords: Newton–Cartan; Milne; hydrodynamics; stress tensor; virial theorem; Joukowski theorem; Bernoulli theorem.

1. Introduction

This paper is the second of a series providing a systematic treatment of the essential dynamical properties of multi-fluid models (as needed for the description of a neutron superfluid moving with respect to a normal background) using a non-relativistic but 4-dimensionally covariant formalism in which the preferred Newtonian time gradient t_μ features as a null eigenvector of the degenerate contravariant space metric tensor $\gamma^{\mu\nu}$. In the preceding paper,[1] which will be referred to simply as (I), it was shown how, in the absence of dissipation, the equations of motion are obtainable, in the conservative limit, from a variation principle given, in a fixed background gravitational field ϕ, by a Lagrangian Λ that is specified as a function of a set of conserved current 4-vectors $n_{\rm X}^{\mu}$. Subject to the condition that the variation of these currents be restricted to have an appropriate world line dependent form, the dynamical equations were derived as the condition of vanishing of a corresponding set of 4-force densities $f_{\nu}^{\rm X}$ that were specified by the equation (I-159) in terms of the dynamical conjugates of the currents, namely the set of 4-covectors $\pi_{\nu}^{\rm X}$ representing the effective energy–momentum per particle of the species labelled by the suffix X.

The present paper (II) describes the generalization needed to allow for active self gravitational effects, which can be dealt with in the conservative case simply by adding an appropriate gravitational field contribution $\Lambda_{\rm grf}$ to the action density. It will be shown how the associated stress momentum energy density 4-tensor will be subject to various (generalized Bernoulli and virial type) conservation laws ensuing from the invariance of the theory with respect not just to Galilean transformations but also to their accelerated, Milne type, generalizations, (which are important for cosmological applications). A formal derivation of the relevant Noether identities is provided in the appendix. A following paper (III) will describe the generalization needed to allow for non-dissipative effects such as viscosity, resistivity and drag on superfluid vortex lines.

2. Stress Momentum Energy Tensor

Experience with the technically simpler relativistic case[2] (and previous work on the Newtonian case[3]) suggests the utility at this stage of introducing the stress momentum energy density tensor

$$T^{\mu}{}_{\nu} = \sum n_{\rm X}^{\mu}\pi_{\nu}^{\rm X} + \Psi\delta^{\mu}_{\nu}\,, \tag{1}$$

in which the covectors $\pi_{\nu}^{\rm X}$ are the 4-momenta defined by (I-152) and Ψ is the generalized pressure function defined by (I-148). The utility of this (chemically covariant) tensor derives from the property that its divergence can be seen to be given just by the sum

$$f_{\mu} = \sum f_{\mu}^{\rm X}\,, \tag{2}$$

of the relevant non-gravitational force densities — which could be expected to cancel each other out, even in a non-conservative model, so long as it is isolated from external interactions — plus the relevant gravitational force density which can always be expected to be present, by the simple formula

$$\nabla_{\mu}T^{\mu}{}_{\nu} = f_{\nu} - \rho\nabla_{\nu}\phi\,, \tag{3}$$

a relation that is shown in the appendix to be interpretable as a Noether identity.

It is to be remarked that the 3-momentum density vector given by (I-146) will be obtainable directly from (1) as the mixed space and time projection specified by

$$\Pi^{\mu} = t_{\rho}T^{\rho}{}_{\nu}\gamma^{\nu\mu}\,. \tag{4}$$

The gauge independent character of this 3-momentum density Π^μ, as manifested by its original specification (I-145) is to be contrasted with the highly gauge dependent character of the corresponding energy density current, namely

$$U^\mu = -T^\mu{}_\nu e^\nu \,, \tag{5}$$

and in particular of its time component, the ordinary energy density, which will be given by

$$U = U^\mu t_\mu = -\sum n_{\rm X} \pi^{\rm X}_\nu e^\nu - \Psi \,. \tag{6}$$

The latter will evidently be decomposable in the form

$$U = U_{\rm mat} + U_{\rm pot} \,, \tag{7}$$

in which the potential energy contribution is just the opposite of the corresponding Lagrangian contribution (I-94), i.e.

$$U_{\rm pot} = \rho\phi = -\Lambda_{\rm pot} \,, \tag{8}$$

while the material contribution will be given by

$$U_{\rm mat} = -\sum n_{\rm X} \mu^{\rm X}_\nu e^\nu - \Psi \,. \tag{9}$$

The complete stress momentum energy density tensor will evidently have a corresponding decomposition as the sum of a purely material part and a gravitational contribution in the form

$$T^\mu{}_\nu = T_{\rm mat}{}^\mu{}_\nu - \phi\rho^\mu t_\nu \,, \tag{10}$$

with

$$T_{\rm mat}{}^\mu{}_\nu = \sum n^\mu_{\rm X} \mu^{\rm X}_\nu + \Psi\delta^\mu_\nu \,, \tag{11}$$

in which Ψ is the same as before, i.e. there is no need to distinguish between Ψ and $\Psi_{\rm mat}$, because its gravitational contribution cancels out so that it is directly expressible as

$$\Psi = \Lambda_{\rm mat} - \sum n^\nu_{\rm X} \mu^{\rm X}_\nu \,, \tag{12}$$

and its general variation (I-151) simplifies as

$$\delta\Psi = -\sum n^\nu_{\rm X}\, \delta\mu^{\rm X}_\nu \,. \tag{13}$$

Actually we can take the decomposition one step further by expressing the material contribution as the sum of a kinetic contribution and a purely internal (and therefore gauge invariant as well as chemically covariant) part in the form

$$T_{\rm mat}{}^\mu{}_\nu = T_{\rm kin}{}^\mu{}_\nu + T_{\rm int}{}^\mu{}_\nu \,, \tag{14}$$

where the kinetic contribution is given simply by

$$T_{\rm kin}{}^\mu{}_\nu = \sum n^\mu_{\rm X} p^{\rm X}_\nu \tag{15}$$

and the internal contribution is the Milne gauge invariant *pressure tensor*, $P^\mu_{\ \nu}$, say, which is given by

$$T_{\text{int}}{}^{\mu}{}_{\nu} = P^\mu_{\ \nu} = \sum n_{\text{x}}^{\ \mu} \chi^{\text{x}}_{\ \nu} + \Psi \delta^\mu_{\ \nu} \,, \tag{16}$$

with Ψ given in terms of quantities obtained just from the internal contribution to the Lagrangian by the formula (I-148). The corresponding decomposition of the material energy contribution (9) will have the form

$$U_{\text{mat}} = U_{\text{kin}} + U_{\text{int}} \tag{17}$$

in which it can be seen, using (I-48), that we shall have

$$U_{\text{kin}} = \Lambda_{\text{kin}} = \tfrac{1}{2} \sum \gamma^\nu_{\ \mu} n_{\text{x}}^{\ \mu} p^{\text{x}}_{\ \nu} \,, \qquad U_{\text{int}} = \sum \gamma^\nu_{\ \mu} n_{\text{x}}^{\ \mu} \chi^{\text{x}}_{\ \nu} - \Lambda_{\text{int}} \,. \tag{18}$$

In terms of the ether rest frame chemical potentials defined by

$$\chi^{\text{x}} = -\chi^{\text{x}}_{\ \nu}\, e^\nu \,, \tag{19}$$

it can be seen that we shall have

$$U_{\text{int}} = \sum n_{\text{x}} \chi^{\text{x}} - \Psi \,, \qquad \delta U_{\text{int}} = \sum \chi^{\text{x}}\, \delta n_{\text{x}} + \sum \gamma^\nu_{\ \mu} n_{\text{x}}^{\ \mu}\, \delta\chi^{\text{x}}_{\ \nu} \,. \tag{20}$$

It follows from identity (I-144) that the contraction of the pressure tensor with the degenerate space metric $\gamma^{\mu\nu}$ gives a result that is symmetric, i.e.

$$\gamma^{\rho[\mu} P^{\nu]}_{\ \ \rho} = 0 \,. \tag{21}$$

It can be seen that this entails a corresponding symmetry property for the space projected part of the complete stress momentum energy density, namely

$$\gamma^\rho_{\ [\mu} \gamma_{\nu]\sigma} T^\sigma_{\ \rho} = 0 \,. \tag{22}$$

Putting all the pieces together again, we see that this complete stress momentum energy density tensor will be expressible as

$$T^\mu_{\ \nu} = P^\mu_{\ \nu} + \sum n_{\text{x}}^{\ \mu} (p^{\text{x}}_{\ \nu} - m^{\text{x}} \phi t_\nu) \,. \tag{23}$$

3. Action and Stress Energy for the Gravitational Field

Up to this point we have been treating the gravitational field just as a given background, but we can promote it to the status of an active dynamical field by taking ϕ to be an extra independent variable in the Lagrangian and adding in an extra gravitational field term so as to obtain a total action density given by

$$\Lambda_{\text{tot}} = \Lambda + \Lambda_{\text{grf}} \,, \tag{24}$$

in which the gravitational field term is given by

$$\Lambda_{\text{grf}} = (8\pi \mathrm{G})^{-1} g^\mu \nabla_\mu \phi \,, \tag{25}$$

where the gravitational field is defined according to the potential ϕ as

$$g^\mu = -\gamma^{\mu\nu} \nabla_\nu \phi \,. \tag{26}$$

Since the only other contribution involving ϕ is the potential energy term $\Lambda_{\rm pot}$ given by (I-136), it can immediately be seen that the requirement of invariance of the total action with respect to localized perturbations of ϕ does indeed give the usual Poisson source equation, which, in the covariant formulation we are using here, will be given by

$$\gamma^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 4\pi \mathrm{G}\rho\,. \tag{27}$$

This can be alternatively presented in a manner reminiscent of the relativistic Einstein equation, using the notation of the Newton–Cartan formalism described above, as

$$R_{\mu\nu} = 4\pi\mathrm{G}\rho t_\mu t_\nu\,, \tag{28}$$

where $R_{\mu\nu}$ is the Ricci type trace of the Newton–Cartan curvature, as given by (I-31).

For reason discussed in the appendix, there will be an associated gravitational stress momentum energy density tensor given by the formula

$$\underset{\rm grf}{T}{}^{\mu}{}_{\nu} = -(4\pi\mathrm{G})^{-1}\big(g^{\mu}\nabla_\nu\phi - \tfrac{1}{2}\delta^{\mu}_{\nu}g^{\rho}\nabla_\rho\phi\big) \tag{29}$$

(which was mistyped in the previous version[3] where the initial minus sign was omitted).

The purely gravitational contribution (29) can be added to the contribution (23) associated with the material source distribution to give a grand total

$$\underset{\rm tot}{T}{}^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} + \underset{\rm grf}{T}{}^{\mu}{}_{\nu}\,, \tag{30}$$

that satisfies the identity

$$\nabla_\mu \underset{\rm tot}{T}{}^{\mu}{}_{\nu} = f_\nu\,, \tag{31}$$

in which all that remains on the right is the sum (2) of the non-gravitational force density contributions, if any, that are defined by the variational formula (I-159).

In the strictly conservative case for which the variational field equations,

$$f^{\rm x}_{\nu} = 0\,, \tag{32}$$

are satisfied, and much more generally in any system that is effectively isolated so that the separate contributions in (32) cancel out in the sum (2), leaving a vanishing total force density, $f_\mu = 0$, we shall simply be left with a total energy momentum conservation law of the form

$$\nabla_\mu \underset{\rm tot}{T}{}^{\mu}{}_{\nu} = 0\,, \tag{33}$$

in which the conserved tensor field (30) can be redecomposed in the form

$$\underset{\rm tot}{T}{}^{\mu}{}_{\nu} = \underset{\rm mat}{T}{}^{\mu}{}_{\nu} + \underset{\rm grt}{T}{}^{\mu}{}_{\nu}\,, \tag{34}$$

where the total gravitational contribution is given by

$$\underset{\rm grt}{T}{}^{\mu}{}_{\nu} = \underset{\rm pot}{T}{}^{\mu}{}_{\nu} + \underset{\rm grf}{T}{}^{\mu}{}_{\nu}\,. \tag{35}$$

It is to be noted that there will be a corresponding total energy density

$$U_{\rm tot} = -t_\mu T_{\rm tot}{}^{\mu}{}_{\nu} e^\nu = U + U_{\rm grf} \tag{36}$$

in which the contribution (29) provides a gravitational field energy density that can be evaluated, using (5), as

$$U_{\rm grf} = -t_\mu T_{\rm grf}{}^{\mu}{}_{\nu} e^\nu = -\Lambda_{\rm grf} = -\tfrac{1}{2}\rho\phi - (8\pi {\rm G})^{-1}\nabla_\mu(\phi g^\mu)\,. \tag{37}$$

It can thereby be seen from (7) that the total energy density will be expressible in the form

$$U_{\rm tot} = U_{\rm mat} + U_{\rm grt} \tag{38}$$

in which the total gravitational energy contribution

$$U_{\rm grt} = U_{\rm pot} + U_{\rm grf} \tag{39}$$

will be given by

$$U_{\rm grt} = -t_\mu T_{\rm grt}{}^{\mu}{}_{\nu} e^\nu = \tfrac{1}{2}\rho\phi - (8\pi {\rm G})^{-1}\nabla_\mu(\phi g^\mu)\,. \tag{40}$$

It is to be observed that the final divergence term in (37) and (40) will cancel itself out when integrated over the volume surrounding a confined source of the kind that will be considered in our discussion of virial theorems in Section 7, provided that (as will always be possible in such a case) the gauge is chosen in such a way that the large distance limit of ϕ is zero. The remaining term in (37) will then half cancel the corresponding potential energy contribution given by (8), leaving a remaining term in (40) that is only half of $U_{\rm pot}$. The effect of this semi-cancellation is to provide a total in which the gravitational binding between each pair of particles is (as it should be) counted only once, not twice.

Another point to be noted is that there will be no corresponding gravitational field contribution to the spacelike 3-momentum density (4), i.e. we shall have

$$t_\rho T_{\rm grf}{}^{\rho}{}_{\nu} \gamma^{\nu\mu} = 0\,. \tag{41}$$

It can thus be seen from (5) that (4) will be replaceable by the equivalent specification

$$\Pi^\mu = t_\rho T_{\rm tot}{}^{\rho}{}_{\nu} \gamma^{\nu\mu}\,. \tag{42}$$

4. Spacetime Symmetries and Homogeneous Cosmological Models

Let us now consider cases in which all or part of the system under study is invariant with respect to the action of one or several symmetry generators, k_a^μ, $a = 0, 1, \ldots$. For example, in an application to a neutron star model, $k_0^{\,\mu}$ might be a stationarity generator, i.e. the generator of a one parameter time translation group (in which case it would specify a natural choice of ether vector by the identification $e^\mu = k_0^{\,\mu}$) and $k_1^{\,\mu}$ might be an axisymmetry generator, i.e. the generator of a one parameter group of rotations about some preferred symmetry axis. This means that all physically well-defined fields must be invariant under the action of the corresponding Lie differentiation operators which we shall designate by the short hand notation $\pounds_a \equiv \vec{k}_a\pounds$, or simply $\pounds = \vec{k}\pounds$ when we are only considering a single generator so that no ambiguity arises. It is to be emphasized however that physical invariance does not necessarily require vanishing of the Lie derivative of a field q that is gauge dependent, but only that its Lie derivative $\pounds q$ should be cancellable by some infinitesimal gauge transformation of the kind denoted by $\breve{\mathrm{d}}q$ in the discussion at the end of Section 6 of Ref. 1. More specifically, all that is required is that for each of the relevant Lie differentiation operators $\pounds_a$ there should be a corresponding infinitesimal gauge transformation operator $\breve{\mathrm{d}}_a$ such that any relevant field satisfies the condition

$$\pounds_a q + \breve{\mathrm{d}}_a q = 0\,. \tag{43}$$

The occurrence of symmetries of this more general kind, involving non-vanishing infinitesimal gauge transformations, is exemplified by the noteworthy case of the Milne type[4] homogeneous cosmological models that are described below.

A minimal requirement for any symmetry generator $k_a^{\,\mu}$ is that its action should preserve the Milne structure of Newtonian spacetime. For the basic Coriolis structure fields t_μ and $\gamma^{\mu\nu}$ the question of whether there is a corresponding gauge transformation operator $\breve{\mathrm{d}}_a$ does not arise, since they are gauge independent, so all that is required is the vanishing of their Lie derivatives, conditions that reduce (I-17) to the form

$$t_\nu \nabla_\mu k_a^{\,\nu} = 0\,, \qquad \gamma^{\rho(\mu}\nabla_\rho k_a^{\,\nu)} = 0\,, \tag{44}$$

from which the space projected and complete spacetime divergence conditions

$$\gamma^\mu_\nu \nabla_\mu k_a^{\,\nu} = 0\,, \qquad \nabla_\nu k_a^{\,\nu} = 0\,, \tag{45}$$

are obtained as corollaries. In a particular Aristotelian frame with respect to an ordinary orthonormal system of coordinates $\{t, X^i\}$, the conditions (44) will be expressible as

$$\nabla_{\!0} k_a^{\,0} = 0\,, \quad \nabla_{\!i} k_a^{\,0} = 0\,, \quad \nabla^{(i} k_a{}^{j)} = 0\,. \tag{46}$$

In the context of general relativity the well-known condition for $k_a^{\,\mu}$ to be a symmetry generator of the spacetime structure is just the well-known Killing equation $\nabla^{(\mu} k_a{}^{\nu)} = 0$, which guarantees the invariance of the spacetime metric $g_{\mu\nu}$ and hence also of the derived Riemannian connection. However in the Newtonian case, since the connection is not simply derivable from the structure fields t_μ and $\gamma^{\mu\nu}$, the corresponding first order differential conditions (44) will not by themselves be sufficient to qualify $k_a^{\,\mu}$ as a spacetime symmetry generator. In order for $k_a^{\,\mu}$ to qualify as a Newtonian spacetime symmetry generator its action must also preserve the flat connection $\Gamma_\mu{}^\nu{}_\rho$. Since the latter is subject to the gauge dependence condition (I-75) its Lie derivative will not have to vanish absolutely, but only modulo the action of some infinitesimal gauge transformation, which will be characterized by some (infinitesimal) boost potential β_a, with a derived time (but not space) dependent boost vector b^μ and a corresponding acceleration vector a^μ, that are given, as in (I-15) by

$$b_a^\mu = \gamma^{\mu\nu}\nabla_\nu\beta_a\,, \qquad \gamma^{\mu\nu}\nabla_\nu b_a^\rho = 0 \tag{47}$$

and

$$a_a^\mu = e^\nu\nabla_\nu b_a^\mu = \gamma^{\mu\nu}\nabla_\nu\alpha_a\,, \qquad \alpha_a = e^\nu\nabla_\nu\beta_a\,. \tag{48}$$

Modulo a choice of rest frame at some arbitrary reference event, the specification, according to (I-16), of an ether frame field e^μ is equivalent to the specification of a corresponding connection $\Gamma_{\mu}{}^{\nu}{}_{\rho}$, namely the one with respect to which the vector field e^μ is mapped onto itself by parallel transport. The preservation of this connection by the action generated by k_a^μ will therefore be ensured simply by the requirement that it should preserve the ether vector. Substituting this vector e^μ in place of q in the general purpose preservation condition (43), one sees from the transformation rule (I-74) and the parallel transport property (I-16) that the requirement of preservation of the ether vector is equivalent to the condition that the relevant boost transformation b_a^μ should be given simply by

$$b_a^\mu = e^\nu\nabla_\nu k_a^\mu\,. \tag{49}$$

It is evident that this will conveniently vanish, so that there will be no need to bother about allowance for the gauge adjustment, in cases for which the symmetry generator k_a^μ is time independent, as will be the case for the symmetries that are most relevant in applications to rotating star models, namely stationarity and axisymmetry. On the other hand the gauge adjustment will have an indispensible role in the kind of time dependent translation symmetries that are relevant in homogeneous self gravitating (cosmological type) configurations such as will be described immediately below. In all cases, it can be seen from (44) that the formula (49) can be used to express the gradient of the symmetry generator as the sum of orthogonally projected and mixed components in the form

$$\nabla_\mu k_a^\nu = \gamma_\mu^\rho\gamma_\sigma^\nu\nabla_\rho k_a^\sigma + t_\mu b_a^\nu\,. \tag{50}$$

In terms of the boost acceleration obtained from (49) using (48), the invariance condition for the connection $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ will be given according to the general principle (43) by the formula

$$\nabla_\mu\nabla_\nu k_a^\rho = t_\mu t_\nu a_a^\rho\,, \tag{51}$$

whose derivation is based on the use of the simple flat space special case of the Yano formula[5] for the Lie derivative of a symmetric connection.

The general version of the Yano formula, including allowance for curvature, is needed for the application to the Newton–Cartan connection (I-25) whose Lie derivative will have the form given by

$$\pounds_a\,\omega_{\mu}{}^{\rho}{}_{\nu} = D_\mu D_\nu k^\rho + R_{\sigma\mu}{}^{\rho}{}_{\nu}k^\sigma\,. \tag{52}$$

In the Newtonian case (unlike the relativistic case) the conditions for invariance of the relevant (Milne) spacetime structure, namely the conditions (44) and (51), are not sufficient to ensure invariance of the gravitational field, as embodied in the independent gauge invariant connection $\omega_\mu{}^\rho{}_\nu$, or equivalently in the gauge dependent field g^μ. In view of its gauge independence, the invariance under the action of k_a^μ of the Newton–Cartan connection requires simply that its Lie derivative, as given above, should vanish,

$$\pounds_a \, \omega_\mu{}^\nu{}_\rho = 0 \,. \tag{53}$$

Subject to the Milne structure invariance conditions (44) and (51), it can be seen from formula (52) (using the expression (I-29) for the curvature) that the supplementary condition (53) for invariance of the gravitational field reduces to the condition obtained by application of the general requirement (43) to the gauge dependent gravitational field vector g^μ. It evidently follows from (I-23) that this invariance requirement will take the form

$$\pounds_a \, g^\mu = a_a^{\,\mu} \,, \tag{54}$$

in which the Lie derivative of the field will of course be given by the well-known commutator formula

$$\pounds_a \, g^\mu = k_a^{\,\nu} \nabla_\nu g^\mu - g^\nu \nabla_\nu k_a^{\,\mu} \,. \tag{55}$$

It similarly follows from (I-24) that at the more highly gauge dependent level of the gravitational potential ϕ, the corresponding field invariance condition will be expressible simply as

$$k_a^{\,\nu} \nabla_\nu \phi = -\alpha_a \,, \tag{56}$$

where α_a is the acceleration potential given by (48).

The prototype illustration of symmetries that are non-trivial, in the sense of requiring a nonzero gauge transformation according to (49), is provided by the case of *homogeneous isotropic* cosmological type models, with expansion described in terms of a comoving radial scale factor σ say, and 3-velocity v^i given in terms of Cartesian space coordinates X^i by $v^i = HX^i$ where the (time dependent but spatially uniform) Hubble parameter is given by $H = \dot{\sigma}/\sigma$ (using a dot for ordinary time derivation). Since the pressure is postulated to be uniform it has no effect on the motion, so the fluid particles will be effectively in free fall with respect to the background gravitational field, which will be given by $\phi = \frac{2}{3}\pi \mathrm{G} \rho \gamma_{ij} X^i X^j$.

One of the things that delayed the discovery of these configurations is their lack of *stationarity*, i.e. their essentially time dependent nature. Their matter distributions, and in particular the density ρ are *not* invariant with respect to the action of the ordinary time translation symmetry generator $k_0^{\,\mu}$ that is definable in terms of the flat Aristotelian coordinates $\{t, X^i\}$ used above by

$$k_0{}^0 = 1 \,, \qquad k_0{}^i = 0 \,, \tag{57}$$

which means that it is simply identifiable with the relevant Aristotelian ether vector,

$$k_0^{\,\mu} = e^\mu \,. \tag{58}$$

The misguided presumption that the cosmological model would have to be stationary was one of the psychological obstacles that, prior to its demolition by Friedmann's theoretical insight (and Hubble's observations) had prevented everyone (including Einstein) from seriously attempting to carry out the non-trivial but (compared with the development of general relativity) not so difficult technical step that was finally taken by Milne.

The stationarity generator (58) illustrates a distinction that does not arise in general relativity theory, where the property of being a Killing vector, i.e. a solution of the equation $\nabla^{(\mu} k_a^{\,\nu)} = 0$, ensures the invariance not only of the background metric $g_{\mu\nu}$ but also, automatically, of the gravitational field as embodied by the associated connection. In a Newtonian context it is necessary to make a distinction between what may be termed weak Killing vectors, or *Killing–Milne vectors*, meaning those that just satisfy the conditions (44) and (51) for invariance of the background spacetime structure, and what may be termed strong Killing vectors or *Killing–Cartan vectors*, meaning those that not only satisfy the conditions (44) and (51) for preservation of the Milne structure but also the supplementary condition (54) for invariance of the gravitational field and hence of the complete Newton–Cartan structure. Regardless of the matter distribution, the stationarity generator $k_0^{\,\mu}$ given by the specification (57) will always be a Killing–Milne vector, since it obviously satisfies (44) and also (with vanishing boost acceleration) (51). However in the particular application described above $k_0^{\,\mu}$ is not a Killing vector in the strong sense because it does not satisfy (54).

Although they are not stationary, the cosmological configurations described above are obviously *isotropic* in the sense of being invariant (in this case without any need for an associated boost transformation) under the action of the set of ordinary angular symmetry generators $k_j^{\,\mu}$ that are definable for values $j = 1, 2, 3$ of the index a as follows. For a rotation about an axis specified by a space unit vector $\nu_j^{\,i}$, with components given, with respect to the flat Aristotelian coordinate system we have been using, simply by

$$\nu_j^{\,i} = \delta_j^{\,i} \,, \tag{59}$$

the corresponding angular symmetry generator will be given by the prescription

$$k_j^{\,0} = 0 \,, \qquad k_j^{\,i} = Y^i{}_k \nu_j^{\,k} \,, \tag{60}$$

in terms[6] of the flat space Killing–Yano 2-form, whose (Cartesian) components are given by

$$Y_{ij} = \varepsilon_{ijk} X^k \,, \tag{61}$$

and which is characterizable, even with respect to non-Cartesian coordinates, by the Killing–Yano equation

$$\nabla_{(i} Y_{j)k} = 0 \,. \tag{62}$$

Whatever the matter distribution may be, it is evident that, like the stationarity generator $k_0^{\,\mu}$, these axisymmetry generators $k_j^{\,\mu}$ for $j = 1, 2, 3$ will always satisfy (44) and (again with vanishing boost acceleration) also (51), so that they will always qualify as Killing vectors in the weak sense. In the particular example of the cosmological application described above it is also evident that (in this case unlike the stationarity generator $k_0^{\,\mu}$) the rotation generators will satisfy the further condition (54) that qualifies them for description as Killing–Cartan vectors.

We now come to the most essential, though not so obvious, symmetry feature of the cosmological configurations described above, which is that as well as being isotropic they are also *homogeneous* (with respect to space but not time) in the sense of being characterized by a set of time dependent space translation symmetries with generators to which we shall attribute negative label values, $a = -1$, -2, -3 (since the positive label values $a = 1, 2, 3$ have already been used up in the specification (60) of the rotation symmetry generators). With respect to the same ordinary Aristotelian system of orthonormal coordinates $\{t, X^i\}$ as above, these space translation symmetry generators will be given by the specification

$$k_{-j}^{\;\;0} = 0\,, \qquad k_{-j}^{\;\;i} = \sigma\delta^i_{\rm j}\,, \tag{63}$$

for $j = 1, 2, 3$. What delayed the discovery of such symmetries (even after Friedmann had overcome the psychological barrier of the stationarity presumption) until Milne saw how to exploit the gauge invariance, is that (in order for the generators (63) to qualify as Killing vectors in both the weak and strong sense described above) the effect of their action needs to be cancelled by the effect of corresponding (non-linearly) time dependent (so non-Galilean) transformations characterized by (spatially uniform) boost vectors $b_{-j}^{\;\;i}$ and boost potentials β_{-j} given by the formulae

$$b_{-j}^{\;\;i} = \dot\sigma\delta^i_j\,, \qquad \beta_{-j} = \dot\sigma\gamma_{ji}X^i\,. \tag{64}$$

In particular the velocity field can immediately be seen to satisfy the invariance condition $\pounds_{-a}v^i = b_{-a}^{\;i}$, while the gravitational field will satisfy the corresponding invariance conditions

$$\pounds_{-j}\, g^i = a_{-j}^{\;\;i}\,, \qquad \pounds_{-j}\,\phi = -\alpha_{-j}\,, \tag{65}$$

(so that the translation generators $k_{-j}^{\;\;\mu}$ qualify as Killing vectors in the strong sense) where

$$a_{-j}^{\;\;i} = \ddot\sigma\delta^{\,i}_j\,, \qquad \alpha_{-j} = \ddot\sigma\gamma_{ji}X^i\,. \tag{66}$$

By contraction with the stress momentum energy density tensor $T^\mu_{\ \nu}$, we can use the symmetry generators $k_a^{\,\mu}$ to construct corresponding generalized momentum currents

$$\mathcal{P}_a^{\,\mu} = T^\mu_{\ \nu}k_a^{\,\nu}\,, \tag{67}$$

of which, for example, $\mathcal{P}_0^{\,\mu}$ will be interpretable as the negative of an ordinary energy current, $\mathcal{P}_1^{\,\mu}$ as a current of ordinary angular momentum (about the X^1 axis in the

Aristotelian coordinate system used above) while $\mathcal{P}_{-1}^{\ \mu}$ will be interpretable as a current transporting a kind of generalized linear momentum (in the direction of the X^1 axis). The latter would reduce to a current of linear momentum of the ordinary kind in the non-expanding (weak gravitational coupling) limit characterized by a time independent value of the factor σ in the specification (63).

Using the Killing vector conditions (44) in the decomposition (50), it can be seen to follow from the stress momentum energy tensor symmetry property (22), that we shall have

$$T^{\mu}_{\ \nu}\nabla_{\mu}k_a^{\ \nu} = T^{\mu}_{\ \nu}t_{\mu}b_a^{\ \nu}\,, \tag{68}$$

where $b_a^{\ \mu}$ is the boost vector given by (49). As a corollary of this useful lemma, it evidently follows from the tensor divergence formula (3) that the corresponding generalized momentum densities will have scalar divergences given by

$$\nabla_{\nu}\mathcal{P}_a^{\ \nu} = k_a^{\ \nu}(f_{\nu} - \rho\nabla_{\nu}\phi) + \Pi^{\nu}\nabla_{\nu}\beta_a\,. \tag{69}$$

On the right of this formula, the first term (involving the total external force f_ν if any) is of a familiar kind, representing the ordinary rate of energy loss (per unit volume) in the case of the stationarity generator $k_0^{\ \mu}$ and representing torque density (about the X^1 axis) in the case of the axisymmetry generator $k_1^{\ \mu}$, and in these cases the last term will be absent. This last term (involving the gradient of the boost potential β_a) will only be needed in cases for which the symmetry generator $k_a^{\ \mu}$ is time dependent. In the case of an isolated system, as characterized by a vanishing total (external) force density, $f_\mu = 0$, and by a total mass current ρ^μ that satisfies the Newtonian mass conservation law (I-108), it can be seen that (69) will be expressible in the form

$$\nabla_{\nu}(\mathcal{P}_a^{\ \nu} - \beta_a\rho^{\nu}) = -\rho(\alpha_a + k_a^{\ \nu}\nabla_{\nu}\phi)\,, \tag{70}$$

in which the right hand side will drop out altogether if the gravitational potential ϕ satisfies the condition (56) of invariance under the symmetry action generated by $k_a^{\ \nu}$, a condition for which it is necessary that $k_a^{\ \nu}$ should not just be a Killing–Milne vector, as has been assumed in the derivation of (69), but more particularly a Killing–Cartan vector in the sense discussed above, meaning that its action preserves the gravitational field as well as the background spacetime structure.

Independently of this latter (field invariance) restriction, but subject to the requirement that the boost contribution should be absent, an isolated system will always be characterized by a conservation law for a corresponding, suitably adjusted, total generalized momentum vector. To start with, we define the (unadjusted) total momentum vector in the obvious way, simply replacing $T^{\mu}_{\ \nu}$ in (67) by the total stress energy tensor (30) which gives

$$\mathcal{P}_{a_{\rm tot}}^{\ \ \mu} = T_{\rm tot\ \nu}^{\ \ \mu}k_a^{\ \nu} = \mathcal{P}_a^{\ \mu} + T_{\rm grf\ \nu}^{\ \ \mu}k_a^{\ \nu}\,. \tag{71}$$

In the case of an isolated system, as characterized by a vanishing total (external) force density, $f_\mu = 0$, and by a total mass current ρ^μ that satisfies the Newtonian mass conservation law (I-108), it can be seen that the analogue of (69) for this total will reduce to a generalized total momentum divergence law of the form

$$\nabla_\nu(\mathcal{P}_{a_{\rm tot}}{}^\nu - \beta_a\rho^\nu) = -\rho\alpha_a\,. \tag{72}$$

Using the Poisson source equation (27) and the identity $\alpha_a\nabla_\nu g^\nu = \nabla_\nu(\alpha_a g^\mu + \phi a_a^{\,\nu})$ the terms can finally be regrouped in a single divergence on the left in the form

$$\nabla_\nu\mathcal{P}_{a_{\rm aug}}{}^\nu = 0\,, \tag{73}$$

expressing conservation of a suitably augmented total generalized momentum current that is given by the prescription

$$\mathcal{P}_{a_{\rm aug}}{}^\nu = \mathcal{P}_{a_{\rm tot}}{}^\nu - \beta_a\rho^\nu - (4\pi\mathrm{G})^{-1}(\alpha_a g^\mu + \phi a_a^{\,\nu})\,. \tag{74}$$

The augmentation contributed by the second, third and fourth terms will actually be needed only in cases for which k_a^μ is time dependent (as in the cosmological example characterized by (63), (64) and (66) as described above). The first term is all that remains, i.e. the unadjusted total $\mathcal{P}_{a_{\rm tot}}{}^\nu$ will be conserved as it stands, in the more familiar cases in which k_a^μ is a generator of ordinary time translations (in which case the conserved vector will be proportional to energy flux) or rotations (for which it will be interpretable as angular momentum flux). It is to be emphasized that the validity of the generalized total momentum conservation law (73) follows just from the postulate that k_a^μ is a Killing–Milne vector in the weak sense discussed above (meaning just that its action preserves the background spacetime structure), so that (unlike the results to be derived in the next section) it will be valid regardless of whether or not the relevant material field configurations have any corresponding symmetry property.

5. Fluid Symmetries and Generalized Bernoulli Theorems

In contexts for which the local gravitational coupling can be considered to be sufficiently weak, it can be of interest to consider configurations of the generic kind governed by (69) in which a material medium, such as the multiconstituent fluid dealt with here, is not necessarily subject to the symmetries of the background spacetime structure and the background gravitational field.

The purpose of the present section is to show how much stronger conclusions can be drawn if the medium is itself invariant under the action of a Killing–Cartan vector k_a^μ of the kind characterized by the spacetime symmetry conditions (44), (45) and the gravitational symmetry condition (54) discussed above.

In the case of a multiconstituent fluid, it can be seen from the formula (I-77) for $\breve{\mathrm{d}}\pi^{\rm x}_\mu$ that in terms of boost potential, the corresponding symmetry conditions on the momentum covectors will be given, according to the general principle (43), by

$$\pounds_a\,\pi^{\rm x}_\nu = m^{\rm x}\nabla_\nu\beta_a\,. \tag{75}$$

Since, by its definition, the Lie derivative of the momentum covector will be given by

$$\pounds_a \pi^{\rm x}_\mu = 2k_a^{\ \nu}\nabla_{[\nu}\pi^{\rm x}{}_{\mu]} + \nabla_\mu\big(k_a^{\ \nu}\pi^{\rm x}{}_\nu\big)\,, \tag{76}$$

we see from definition (I-161) of the vorticity that the symmetry condition on the momentum will be expressible in the form

$$\varpi^{\rm x}_{\mu\nu}k_a^{\ \nu} = \nabla_\mu \mathcal{B}^{\rm x}_a\,, \tag{77}$$

in which $\mathcal{B}^{\rm x}_a$ is a generalized momentum component that is defined by the formula

$$\mathcal{B}^{\rm x}_a = k_a^{\ \nu}\pi^{\rm x}_\nu - m^{\rm x}\beta_a \tag{78}$$

and that can be identified as a generalization of the historic Bernoulli scalar.

It is immediately obvious from (77) that we obtain a strong generalization of the Bernoulli theorem to the effect that if the flow of some particular constituent is irrotational, then the corresponding Bernoulli scalar will be uniform throughout, in the sense of being independent of both space and time,

$$\varpi^{\rm x}_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla_\nu \mathcal{B}^{\rm x}_a = 0\,. \tag{79}$$

In the case (of the kind originally considered by Bernoulli) of ordinary time translation symmetry, as obtained by setting $a = 0$ with the specification (57), the corresponding value,

$$\mathcal{B}^{\rm x}_0 = \pi^{\rm x}_\nu e^\nu\,, \tag{80}$$

will be interpretable as the negative of an effective energy per particle. Similarly in the case of an ordinary rotation symmetry, as obtained, for example by setting $a = 1$ (for a rotation about the X^1 axis) according to the specification (60) the corresponding value,

$$\mathcal{B}^{\rm x}_1 = \pi^{\rm x}_\nu k_1^\nu\,, \tag{81}$$

will be interpretable as an effective angular momentum per particle. In the case of a boosted translation symmetry along for example the X^1 axis, as obtained by setting $a = -1$ in the specification (63), we obtain a generalized Bernoulli scalar

$$\mathcal{B}^{\rm x}_{-1} = k_{-1}^{\ \nu}\pi^{\rm x}_\nu - m^{\rm x}\beta_{-1}\,, \tag{82}$$

of a new kind, whose interpretation is not so elementary, due to the involvement of the boost potential β_{-1}.

In the general case, involving rotation or even transfer of matter between different constituents, it can be seen by combining (77) with the general force density formula (I-164) that the gradient of the relevant generalized Bernoulli scalar along the flow lines will be given by

$$n_{\rm x}^{\ \nu}\nabla_\nu \mathcal{B}^{\rm x}_a = k_a^{\ \nu} f^{\rm x}_\nu + k_a^{\ \nu}\pi^{\rm x}_\nu \mathcal{D}_{\rm x}\,. \tag{83}$$

We thus obtain a generalized weak Bernoulli theorem to the effect that in the conservative case of vanishing force density, the Bernoulli scalar will be constant along each separate flow line,

$$f^{\rm x}_{\nu} = 0\,, \qquad \mathcal{D}_{\rm x} = 0 \;\;\Rightarrow\;\; n_{\rm x}^{\ \nu}\nabla_{\nu}\mathcal{B}^{\rm x}_{a} = 0\,. \tag{84}$$

A useful alternative presentation of this generalized Bernoulli theorem can be obtained by introducing generalized momentum current densities that are defined for each separate constituent by the prescription

$$\mathcal{P}^{{\rm x}\,\mu}_{a} = \mathcal{B}^{\rm x}_{a} n_{\rm x}^{\ \mu}\,. \tag{85}$$

The theorem (83) can thereby be reformulated as a divergence law of the form

$$\nabla_{\nu}\mathcal{P}^{{\rm x}\,\nu}_{a} = k_{a}^{\ \nu} f^{\rm x}_{\nu} + \beta_{a} m^{\rm x}\mathcal{D}_{\rm x}\,. \tag{86}$$

It can be seen that the generalized total (non-gravitational) momentum density (67) is related to the sum of the separate constituent contributions by

$$\mathcal{P}_{a}^{\ \mu} - \beta_{a}\rho^{\mu} = \sum \mathcal{P}^{{\rm x}\,\mu}_{a} + \Psi k_{a}^{\ \mu}\,. \tag{87}$$

It was already made apparent by (71) in the preceding section that the combination on the left of this equation will be conserved under rather general circumstances: all that is required is that the system as a whole should be effectively isolated from external influences (other than gravity) and that the generator $k_{a}^{\ \mu}$ should be a Killing–Cartan vector in the sense specified above. What has been shown in the present section is that if we make the more restrictive postulate that the material system itself is invariant, in the gauge adjusted sense characterized by (43), under the action generated by the Killing–Cartan vector, and if we also postulate that the separate force density contributions $f^{\rm x}_{\mu}$ (and hence also the separate decay rates $\mathcal{D}_{\rm x}$) all vanish individually, then (86) will be applicable so as to provide the much stronger conclusion that *each* of the distinct contributions in the sum on the right of (87) will be *separately* conserved.

6. Generalized Joukowski Theorem

As an application of the generalized Bernoulli theorem discussed in the previous section, the Joukowski formula[7] is extended to the case of multiconstituent fluid. We shall derive the Magnus force arising on a perturbing vortex moving in an *asymptotically uniform* medium characterized by vanishing currents $\overline{n_{\rm x}}^{\ \nu} = 0$ (asymptotic values will be denoted by an overhead bar). The gravitational potential is supposed to be unaffected by the perturbation. We shall follow closely a previous analysis in a relativistic context.[8] The fluid flow is supposed to be stationary thereby admitting the ether vector e^{μ} as a Cartan–Killing vector, and longitudinally invariant along the uniform spacelike Cartan–Killing vector l^{μ} (vortex symmetry axis) satisfying $l^{\mu}t_{\mu} = 0$.

The multiconstituent flow is described by the conservation law (33) $\nabla_{\mu}T^{\ \mu}_{{\rm tot}\,\nu} = 0$. Besides each of the currents are separately conserved $\nabla_{\nu}n_{\rm x}^{\ \nu} = 0$. It is further assumed that not only the total force density vanishes, but that each fluid component is isolated $f^{\rm x}_{\mu} = n_{\rm x}^{\nu}\varpi^{\rm x}{}_{\nu\mu} = 0$. It thus implies that the generalized vorticity 2-form $\varpi^{\rm x}{}_{\nu\mu}$ will vanish whenever it is initially zero $\varpi^{\rm x}{}_{\nu\mu} = 0$. The framework in which the Joukowski formula is derived, is restricted to such *irrotational* flows (whether it is superfluid or not).

Consequently, from (79), the Bernoulli scalars associated with the corresponding Killing vectors are uniform $\nabla_{\nu}\mathcal{B}^{\rm x}_{a} = 0$. In particular, the following Bernoulli scalars are constants:

$$\mathcal{B}^{\rm x}_{0} = \pi^{\rm x}_{\nu}e^{\nu} = \overline{\pi^{\rm x}}_{\nu}e^{\nu}\,, \tag{88}$$

$$\mathcal{B}^{\rm x}_{-1} = \pi^{\rm x}_{\nu}l^{\nu} = \overline{\pi^{\rm x}}_{\nu}l^{\nu}\,. \tag{89}$$

The multiconstituent fluid is exerting a force (per unit length) on the vortex given by

$$\mathcal{F}_{\nu} = \oint T_{_{\rm tot}}{}^{\sigma}_{\ \nu}\, {}^{\star}\varepsilon_{\sigma\mu}dx^{\mu}\,, \tag{90}$$

in which the "background" antisymmetric measure tensor is defined by

$${}^{\star}\varepsilon_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}e^{\rho}l^{\sigma}\,. \tag{91}$$

The conservation of the total stress energy momentum tensor allows one to evaluate the integral along any closed circuit, which for convenience can be chosen sufficiently far away from the vortex core for a linear expansion to be valid:

$$T_{_{\rm tot}}{}^{\sigma}_{\ \nu} = \overline{T_{_{\rm tot}}}{}^{\sigma}_{\ \nu} + \delta T_{_{\rm tot}}{}^{\sigma}_{\ \nu} + O(\delta^2)\,. \tag{92}$$

The force is zero by definition in the unperturbed uniform medium hence it reduces to

$$\mathcal{F}_{\nu} = \delta\mathcal{F}_{\nu} + O(\delta^2)\,. \tag{93}$$

The linear deviation of the stress energy momentum tensor from the uniform background value is expressible as

$$\delta T_{_{\rm tot}}{}^{\sigma}_{\ \nu} = \sum\left(\overline{\pi^{\rm x}}_{\nu}\,\delta n_{\rm x}^{\sigma} + \overline{n_{\rm x}}^{\sigma}\,\delta\pi^{\rm x}_{\nu} - \overline{n_{\rm x}}^{\rho}\,\delta\pi^{\rm x}_{\rho}\,\delta^{\sigma}_{\nu}\right). \tag{94}$$

The force per unit length to lowest order is therefore given by $\delta\mathcal{F}_{\nu} = \oint \delta T_{_{\rm tot}}{}^{\sigma}_{\ \nu}\, {}^{\star}\varepsilon_{\sigma\mu}dx^{\mu}$, i.e.

$$\delta\mathcal{F}_{\nu} = \sum\left(\overline{\pi^{\rm x}}_{\nu}\oint \delta n_{\rm x}^{\sigma}\, {}^{\star}\varepsilon_{\sigma\mu}dx^{\mu} + 2\overline{n_{\rm x}}^{\sigma}\oint \delta\pi^{\rm x}{}_{[\nu}\, {}^{\star}\varepsilon_{\sigma]\mu}dx^{\mu}\right). \tag{95}$$

It is then worthwhile to notice as a consequence of the Bernoulli theorem that the variation of the momentum is purely orthogonal to the vortex

$$\perp^{\sigma}_{\nu}\,\delta\pi^{\rm x}_{\sigma} = \delta\pi^{\rm x}_{\nu}\,, \tag{96}$$

where $\perp^{\sigma}_{\nu}$ is the operator of projection orthogonal to the vortex (i.e. orthogonal to the Killing vectors). The spacetime metric δ^{σ}_{ν} is decomposible into the sum of the operators of projection parallel η^{σ}_{ν} and orthogonal $\perp^{\sigma}_{\nu}$ to the vortex. The contravariant antisymmetric measure tensor ${}^{\star}\varepsilon^{\mu\nu}$ is introduced by

$$ {}^{\star}\varepsilon^{\mu\nu} = t_{\rho} l_{\sigma} \varepsilon^{\mu\nu\rho\sigma} \,. \tag{97} $$

The covariant component l_{σ} of the Killing vector is well-defined since this vector is spacelike. The projection operators are given by

$$ \eta^{\sigma}_{\nu} = e^{\sigma} t_{\nu} + l^{\sigma} l_{\nu} \,, \tag{98} $$

$$ \perp^{\sigma}_{\nu} = -{}^{\star}\varepsilon^{\mu\sigma}\, {}^{\star}\varepsilon_{\mu\nu} \,. \tag{99} $$

The force per unit length therefore becomes

$$ \delta\mathcal{F}_{\nu} = \sum \left(\overline{\pi^{\mathrm{X}}}_{\nu} \oint \delta n^{\sigma}_{\mathrm{X}}\, {}^{\star}\varepsilon_{\sigma\mu} dx^{\mu} + 2\overline{n_{\mathrm{X}}}^{\sigma} \oint \delta\pi^{\mathrm{X}}_{\rho} \perp^{\rho}_{[\nu}\, {}^{\star}\varepsilon_{\sigma]\mu} dx^{\mu} \right) . \tag{100} $$

Using the following identity,

$$ \perp^{\rho}{}_{[\nu}\, {}^{\star}\varepsilon_{\sigma]\mu} = -{}^{\star}\varepsilon_{\nu\sigma} \perp^{\rho}_{\mu} \,, \tag{101} $$

the force per unit length acting on the vortex simplifies to

$$ \delta\mathcal{F}_{\nu} = \sum \left(\overline{\pi^{\mathrm{X}}}_{\nu} \delta\mathcal{D}_{\mathrm{X}} + \overline{n_{\mathrm{X}}}^{\sigma}\, {}^{\star}\varepsilon_{\sigma\nu} \delta\mathcal{C}^{\mathrm{X}} \right) , \tag{102} $$

in which the momentum circulation integral $\mathcal{C}^{\mathrm{X}}$ and the current outflux integral $\mathcal{D}_{\mathrm{X}}$ are defined by

$$ \mathcal{D}_{\mathrm{X}} = \oint n^{\sigma}_{\mathrm{X}}\, {}^{\star}\varepsilon_{\sigma\mu} dx^{\mu} \,, \tag{103} $$

$$ \mathcal{C}^{\mathrm{X}} = \oint \pi^{\mathrm{X}}_{\nu}\, dx^{\nu} \,. \tag{104} $$

The current and vorticity conservation laws ensure that $\mathcal{C}^{\mathrm{X}}$ and $\mathcal{D}_{\mathrm{X}}$ respectively, do not depend on the integration closed path. Therefore the circuit can be chosen to lie at a very large distance from the vortex core so that the force is exactly given by the linear perturbation term. In cases for which there is no current creation in the vortex core, the current outflux integral will simply vanish $\mathcal{D}_{\mathrm{X}} = 0$ and since the asymptotic values of these two integrals must also be equal to zero the force per unit length acting on the vortex is eventually given by

$$ \mathcal{F}_{\nu} = {}^{\star}\varepsilon_{\sigma\nu} \sum \mathcal{C}^{\mathrm{X}} \overline{n_{\mathrm{X}}}^{\sigma} \,. \tag{105} $$

7. Virial Moment Theorems for Isolated System

The preceding analysis has been essentially local, but of course whenever one is concerned with a system — such as a star model — that is confined within a compact region it is of particular interest to consider global quantities, particularly those that are subjected to simple evolution equations or are actually conserved.

A specially noteworthy example is of course the total energy $E_{\rm tot}$ which is definable as a function of time by a space integral that is expressible in the form

$$E_{\rm tot} = \int U_{\rm tot}\, {\rm d}^3 X \,, \tag{106}$$

with respect to standard Aristotelian coordinates $\{X^{\rm o}, X^i\}$ of the kind described in Section 2 of Ref. 1, with time coordinate $X^{\rm o} = t$, and space coordinates X^i of ordinary Cartesian type, so that the 3-dimensional Euclidean metric components γ_{ij} are those of a 3 by 3 unit matrix, while the time covector has components $t_{_0} = 1$, $t_i = 0$, and the corresponding ether velocity vector has components $e^{\,\rm o} = 1$, $e^i = 0$.

It is evident from the local energy momentum conservation law (33) that the time evolution of this quantity can be expressible as the space integral of a divergence in the form

$$\frac{\rm d}{{\rm d}t}\, E_{\rm tot} = \int \nabla_{\!i}\, T_{\!{\rm tot}\,0}^{\ i}\, {\rm d}^3 X \,. \tag{107}$$

Using Green's theorem to convert this to an asymptotic surface integral at large distance, it can be seen to follow that for an isolated system the energy will actually be conserved, i.e. we shall simply have

$$\frac{\rm d}{{\rm d}t} E_{\rm tot} = 0 \,, \tag{108}$$

provided that (as is always possible in such a case) the gauge for the gravitational potential ϕ in the formula (29) for the external field contribution $T_{\!{\rm grf}\,0}^{\ i}$ is chosen in accordance with the usual convention that it should tend to zero at large distances. (This condition automatically entails that ϕ will fall off as the inverse of the radial distance, a requirement which is sufficient to get rid of the boundary term that would otherwise be left over.)

The foregoing total energy conservation law depends only on the time projected part of the complete local energy momentum conservation law (33), which can be decomposed in terms of separate energy density and 3-momentum density components

$$U_{\rm tot} = -T_{\!{\rm tot}\,0}^{\ 0} \,, \qquad \mathit{\Pi}^i = T_{\!\rm tot}^{\ 0i} = \gamma^{ik} T_{\!{\rm tot}\,k}^{\ 0} \,, \tag{109}$$

in the form

$$\nabla_{\!_0} U_{\rm tot} = \nabla_{\!i} T_{\!{\rm tot}\,0}^{\ i} \,, \tag{110}$$

(the part from which (107) is derived) and

$$\nabla_{\!_0} \mathit{\Pi}^i = -\nabla_{\!j} T_{\!\rm tot}^{\ ji} = -\gamma^{ik} \nabla_{\!j} T_{\!{\rm tot}\,k}^{\ j} \,. \tag{111}$$

(in which, as remarked above there is no need to distinguish between $\mathit{\Pi}^i$ and $\mathit{\Pi}_{\rm tot}^{\,i}$ since there is no purely gravitational 3-momentum contribution). The space projected part, namely the local 3-momentum conservation law (111) is particularly informative when used in conjunction with the conservation law (I-108) for the Newtonian mass current, whose time and space components

$$\rho^0 = \rho\,, \qquad \rho^i = \Pi^i \tag{112}$$

must satisfy

$$\nabla_{\!0}\rho = -\nabla_{\!i}\,\rho^i\,. \tag{113}$$

By combining this with (111) we obtain a noteworthy second order differential relation,

$$\nabla_{\!0}\nabla_{\!0}\rho = \nabla_{\!i}\nabla_{\!j}T^{ij}_{\text{tot}}\,, \tag{114}$$

that may appropriately be referred to as the local Newtonian virial equation, since, as will be shown below, it is what ultimately accounts for various older (specialized) and newer (more general) variants of what is commonly referred to by the term “virial theorem.”

The relevant global applications of the foregoing local conservation laws involve global mass–moment integrals of the form

$$\mathcal{M} = \int \rho\,\mathcal{X}\,\mathrm{d}^3X \tag{115}$$

and momentum–moment integrals of the form

$$\mathcal{J} = \int \Pi^i\,\mathcal{Y}_i\,\mathrm{d}^3X\,, \tag{116}$$

where $\mathcal{X}$ and $\mathcal{Y}_i$ are weighting factors that are constructed as fixed (i.e. time independent) given functions of the Cartesian space coordinates X^i. In the same way as (108) was obtained by applying (110) to (106) using Green's theorem to get rid of a divergence, it can be seen that — for a locally confined system — application of (113) in (115) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} = \int \Pi^i\nabla_{\!i}\mathcal{X}\,\mathrm{d}^3X\,. \tag{117}$$

In order for application of (111) in (116) to provide the analogous relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J} = \int T^{ij}_{\text{tot}}\nabla_{\!i}\mathcal{Y}_j\,\mathrm{d}^3X\,, \tag{118}$$

it is not enough to require that the material system be locally confined: in order for Green's theorem to be able to get rid of the relevant divergence contribution the weighting factor $\mathcal{Y}_i$ must satisfy an appropriate admissibility restriction. Since, according to (29), the components of the gravitational field contribution T^{ij}_{grf} will fall off as the fourth power of the radial distance from the material source, it can be seen that the criterion for admissibility in (118) is that the components $\mathcal{Y}_i$ should grow more slowly than the square of the radial distance. Subject to the proviso that $\mathcal{X}$ should satisfy this admissibility condition when one makes the substitution $\mathcal{Y}_i = \nabla_{\!i}\mathcal{X}$, it can be seen by combining (117) and (118) — or directly from the local virial relation (114) — that we shall obtain a generic global virial relation of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{M} = \int \underset{\text{tot}}{T}{}^{ij}\nabla_i\nabla_j\mathcal{X}\,\mathrm{d}^3X\,. \tag{119}$$

The simplest example of the application of the foregoing formulae is of course to the case of the total mass M, which is obtained simply by taking the weighting factor $\mathcal{X}$ to be unity. Thus by setting

$$\mathcal{X} \mapsto 1 \quad \Rightarrow \quad \mathcal{M} \mapsto M\,, \qquad \nabla_i\mathcal{X} \mapsto 0\,, \tag{120}$$

we see that (115) and (117) reduce respectively to the definition and conservation law for the total mass as given by

$$M = \int \rho\,\mathrm{d}^3X\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}M = 0\,. \tag{121}$$

The next simplest possibility is the case of the dipole moment, which is obtained by taking the weighting factor to be linearly dependent on the Cartesian space coordinates X^i. Thus in particular the dipole moment in the direction of, say, the X^{3} axis is obtained by just taking the weighting factor $\mathcal{X}$ to be X^{3}. Thus by setting

$$\mathcal{X} \mapsto X^{3} \quad \Rightarrow \quad \mathcal{M} \mapsto \mathcal{D}^{3}\,, \qquad \nabla_i\mathcal{X} \mapsto \gamma^{3}_{i}\,, \qquad \nabla_i\nabla_j\mathcal{X} \mapsto 0\,, \tag{122}$$

we see that (115) and (117) reduce respectively to the definition and variation law for the dipole moment component $\mathcal{D}^{3}$ as given by

$$\mathcal{D}^{3} = \int \rho\,X^{3}\,\mathrm{d}^3X\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}^{3} = \int \Pi^{3}\,\mathrm{d}^3X\,, \tag{123}$$

where the latter integral is interpretable as the ordinary linear 3-momentum in the direction of the X^{3} axis, whose conservation is now given by the relevant application of (118), or equivalently of (119), which can be seen to reduce simply to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{D}^{3} = 0\,. \tag{124}$$

This particular application will of course reduce to a triviality if we exploit the freedom to use the centre of mass frame characterized by the condition that the dipole moment simply vanishes.

A less trivial application is to the case of angular momentum, whose components are obtained by taking the weighting factor $\mathcal{Y}_i$ to be given by the corresponding components of the Killing–Yano 2-form (61). Thus in particular the angular momentum J_{3} about the X^{3} axis will be obtained by identifying the weighting components $\mathcal{Y}_i$ with the Killing–Yano components $Y_{i_3} = \varepsilon_{i_3 k}X^{k}$. Thus by setting

$$\mathcal{Y}_i \mapsto Y_{i_3} \quad \Rightarrow \quad \mathcal{J} \mapsto J_{3}\,, \qquad \nabla_i\mathcal{Y}_j \mapsto \varepsilon_{ij_3}\,, \tag{125}$$

we see that (116) and (118) reduce respectively to the definition and the conservation law for this angular momentum component, as given by

$$J_{3} = \varepsilon_{ij_3}\int X^i\,\Pi^j\,\mathrm{d}^3X\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}J_{3} = 0\,, \tag{126}$$

in which the last step, namely the vanishing of the time derivative, results from the stress tensor symmetry property

$$T_{\rm tot}^{\,[ij]} = 0\,, \tag{127}$$

which holds as a consequence of the corresponding property (22) of the material contribution, and of the manifest symmetry of the gravitational contribution (29). The familiar conclusion that the total angular momentum of an isolated Newtonian system will be conserved can alternatively be interpreted as an immediate global consequence, via Green's theorem, of the relevant particular application of the general local conservation law (74).

Having seen how mass and angular momentum conservation laws of the usual kind can immediately be recovered as particularly simple special cases for which the weighting factor is uniform or linearly dependent on the Cartesian space coordinates — so that the term on the right of (119) will drop out — we now come to the less trivial category of applications for which the term "virial theorem" is most commonly employed,[9] namely cases for which the weighting factor $\mathcal{X}$ has homogeneous quadratic dependence on the Cartesian space coordinates X^i. The simplest such possibility is the isotropic case obtained by taking $\mathcal{X}$ proportional to the square of the radial distance r from the center as given (in a Cartesian system with origin at the center of mass) by

$$r^2 = \gamma_{ij} X^i X^j \tag{128}$$

while another obviously important special case is that for which $\mathcal{X}$ is taken to be proportional to the square of the distance ϖ from, say, the X^3 axis, as defined by

$$\varpi^2 = \big(\gamma_{ij} - \delta^3_i \delta^3_j\big) X^i X^j = (X^1)^2 + (X^2)^2\,. \tag{129}$$

The isotropic case obtained simply by identifying $\mathcal{X}$ with r^2 is that for which $\mathcal{M}$ is just the ordinary scalar quadrupole moment I. Thus by setting

$$\mathcal{X} \mapsto r^2 \quad\Rightarrow\quad \mathcal{M} \mapsto Q\,, \qquad \nabla_i \nabla_j \mathcal{X} \mapsto 2\gamma_{ij}\,, \tag{130}$$

one sees from the generic virial relation (119) that the scalar quadrupole moment

$$Q = \int \rho\, r^2\, \mathrm{d}^3 X \tag{131}$$

will satisfy a time evolution equation of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Q = 2 \int T_{\rm tot}{}^{i}{}_{i}\, \mathrm{d}^3 X\,. \tag{132}$$

It evidently follows that for a stationary (i.e. time independent) configuration the integral on the right of this equation must vanish.

The axial case obtained by identifying $\mathcal{X}$ with ϖ^2 is that for which $\mathcal{M}$ is just the moment of inertia moment I_3 about the X^3 axis. Thus by setting

$$\mathcal{X} \mapsto \varpi^2 \quad\Rightarrow\quad \mathcal{M} \mapsto I_3\,, \qquad \nabla_i \nabla_j \mathcal{X} \mapsto 2\big(\gamma_{ij} - \delta^3_i \delta^3_j\big)\,, \tag{133}$$

one sees from the generic virial relation (119) that this moment of inertia

$$I_3 = \int \rho\, \varpi^2\, \mathrm{d}^3X \tag{134}$$

will satisfy a time evolution equation of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_3 = 2 \int \left(T_{\mathrm{tot}}^{\ 11} + T_{\mathrm{tot}}^{\ 22} \right) \mathrm{d}^3X \,. \tag{135}$$

It evidently follows again that for a stationary (i.e. time independent) configuration the integral on the right of this equation must vanish.

The preceding special isotropic and axial cases can be considered as combinations of the separate components of the corresponding tensorial theorem — as obtained by substituting $X^i X^j$ in place of $\mathcal{X}$ in (117) and (119) — according to which the first and second time derivatives of the generic quadrupole moment component

$$Q^{ij} = \int \rho\, X^i X^j\, \mathrm{d}^3X \tag{136}$$

will satisfy equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} Q^{ij} = 2 \int \Pi^{(i} X^{j)}\, \mathrm{d}^3X \tag{137}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Q^{ij} = 2 \int T_{\mathrm{tot}}^{\ ij}\, \mathrm{d}^3X \,. \tag{138}$$

Just like the stationary case, for which the integrals on the right of (137) and (138) must vanish, the time dependent case is also of particular interest because it is the third time derivative of the trace free part of the quadrupole moment tensor that provides the source of gravitational radiation in the Newtonian (weak field, low velocity) limit of general relativity.

The foregoing conclusions, and in particular the scalar quadrupole evolution equation (132), can be construed as a concise general statement of what is commonly referred to — in more detailed applications to particular well-known cases — as the "virial theorem". The most widely familiar version[9] applies just to the case of a single constituent perfect fluid, but in a recent study of ellipsoidal configurations Sedrakian and Wasserman[10] have provided a version applying to cases in which there is a pair of independently moving fluid constituents. Our present — formally very simple — result (132) actually goes further, not just by allowing for the possibility of more than two constituents, but also, less trivially, by including allowance for the effect of entrainment whereby the strong coupling between the constituents modifies the momenta of the constituent particles whose effective masses may deviate substantially from their ordinary rest masses.

To relate our concise new general purpose "virial theorem" (132) to the more specialized results that are already well-known, it is instructive to consider the distinct contributions that are involved. As as start, it can be seen from (38) that the total energy (106) can be expressible in the form

$$E_{\rm tot} = E_{\rm mat} + E_{\rm grt} \,, \tag{139}$$

where the purely material contribution is defined by

$$E_{\rm mat} = \int U_{\rm mat}\, {\rm d}^3 X \,, \tag{140}$$

and the total gravitational binding energy contribution is defined by

$$E_{\rm grt} = E_{\rm grf} + E_{\rm pot} = \int U_{\rm grt}\, {\rm d}^3 X \,, \tag{141}$$

in which

$$E_{\rm grf} = \int U_{\rm grf}\, {\rm d}^3 X = \frac{1}{8\pi {\rm G}} \int g^i g_i\, {\rm d}^3 X \,, \tag{142}$$

and

$$E_{\rm pot} = \int U_{\rm pot}\, {\rm d}^3 X = \int \phi\rho\, {\rm d}^3 X \,. \tag{143}$$

For a confined source, it can be seen from (40) using Green's theorem that the total gravitational contribution (141) will be related to the separate gravitational field contribution (142) and gravitational potential contribution (143) by

$$E_{\rm grt} = -E_{\rm grf} = \tfrac{1}{2} E_{\rm pot} \,. \tag{144}$$

In a similar manner, using the observation that

$$T_{\rm grt}{}^{i}{}_{i} = -U_{\rm grf} \,, \tag{145}$$

it can be seen that the integral on the right hand side of (119) can be split as a sum of a purely material contribution and a gravitational contribution, in which (again using Green's theorem) the latter works out to be the same — for a confined source — as the corresponding contribution to the energy, i.e. one obtains

$$\int T_{\rm tot}{}^{i}{}_{i}\, {\rm d}^3 X = \int T_{\rm mat}{}^{i}{}_{i}\, {\rm d}^3 X + E_{\rm grt} \,. \tag{146}$$

The material energy contribution can evidently be further decomposed as

$$E_{\rm mat} = E_{\rm kin} + E_{\rm int} \,, \tag{147}$$

where the separate kinetic and internal contributions are expressible in terms of the densities given by (18) as

$$E_{\rm kin} = \int U_{\rm kin}\, {\rm d}^3 X \,, \qquad E_{\rm int} = \int U_{\rm int}\, {\rm d}^3 X \tag{148}$$

of which the latter will be expressible in terms of the rest frame chemical potentials $\chi^{\rm x}$ introduced in (19) as

$$E_{\rm int} = \sum \int n_{\rm x} \chi^{\rm x}\, {\rm d}^3 X - \int \Psi\, {\rm d}^3 X \,. \tag{149}$$

The corresponding decomposition for the material contribution in (145) has the form

$$\int T_{\mathrm{mat}}{}^{i}{}_{i}\,\mathrm{d}^3X = \int T_{\mathrm{kin}}{}^{i}{}_{i}\,\mathrm{d}^3X + \int T_{\mathrm{int}}{}^{i}{}_{i}\,\mathrm{d}^3X\,, \tag{150}$$

in which the kinetic contribution can be seen from (15) and (18) to be given by

$$\int T_{\mathrm{kin}}{}^{i}{}_{i}\,\mathrm{d}^3X = 2E_{\mathrm{kin}}\,, \tag{151}$$

while the internal contribution can be seen from (16) to be given by

$$\int T_{\mathrm{int}}{}^{i}{}_{i}\,\mathrm{d}^3X = 3\int \Psi\,\mathrm{d}^3X + \sum \int n_{\mathrm{X}}^{\,i}\chi^{\mathrm{X}}_{i}\,\mathrm{d}^3X\,. \tag{152}$$

Putting the separate pieces together, it can be seen that the integral in the formulation (132) of the generalized "scalar virial theorem" — and that in any stationary equilibrium configuration must therefore vanish, i.e.

$$\int T_{\mathrm{tot}}{}^{i}{}_{i}\,\mathrm{d}^3X = 0\,, \tag{153}$$

— will be expressible by

$$\int T_{\mathrm{tot}}{}^{i}{}_{i}\,\mathrm{d}^3X = E_{\mathrm{grt}} + 2E_{\mathrm{kin}} + 3\int \Psi\,\mathrm{d}^3X + \sum \int n_{\mathrm{X}}^{\,i}\chi^{\mathrm{X}}_{i}\,\mathrm{d}^3X\,. \tag{154}$$

In the combination on the right hand side — which must vanish for a stationary equilibrium configuration — it is to be remarked that the first two terms (the total gravitational energy plus twice the ordinary kinetic energy) are of the kind that is traditionally familiar. As compared with the multiconstituent version given recently by Sedrakian and Wasserman[10] for the idealized case in which mutual entrainment between the constituents was ignored, the difference here consists firstly of the use of the undecomposable pressure function Ψ in place of a sum (of the form $\sum P_{\mathrm{X}}$) of ordinary pressure contributions from the distinct constituents, and secondly of the inclusion of the final term involving the previously ignored entrainment momenta χ^{X}_{i} themselves.

Before concluding, we wish to point out that the homogeneously quadratic category that has been discussed in detail immediately above is not the only useful category of non-trivial applications of our generic virial theorem (119). Another category worth consideration is that of homogeneously first (as opposed to second) order mass moments. Strictly linear dependence on the Cartesian space coordinates leads merely to the trivial dipole moment case discussed above, but a no less natural, and much more interesting, alternative possibility is that of the isotropic homogeneously first order (but non-linear) case obtained by setting $\mathcal{X} = r$, while the next most obviously interesting possibility is that of the cylindrical homogeneously first order (but non-linear) case obtained by setting $\mathcal{X} = \varpi$.

For the first of these, namely the isotropic first order case, we have

$$\mathcal{X} = r \quad\Rightarrow\quad \nabla_i\nabla_j\mathcal{X} = r^{-1}(\gamma_{ij} - \nu_{ri}\nu_{rj})\,, \tag{155}$$

where $\nu_r^{\,i}$ is the radial unit vector as defined by

$$\nu_r^{\,i} = r^{-1} X^i \,. \tag{156}$$

Since, in terms of standard spherical coordinates r, θ, φ, the volume element will be given by $\mathrm{d}X^1\, \mathrm{d}X^2\, \mathrm{d}X^3 = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi$, the isotropic homogeneously linear virial theorem obtained by substituting (155) in (119) will be expressible as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \rho\, r^3 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi = \int \big(\underset{\rm tot}{T}{}^{i}{}_{i} - \underset{\rm tot}{T}{}^{ij} \nu_{r\,i} \nu_{r\,j} \big) r \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi \,. \tag{157}$$

An even simpler relation is obtainable for the cylindrical first order case, for which we have

$$\mathcal{X} = \varpi \quad \Rightarrow \quad \nabla_i \nabla_j \mathcal{X} = \varpi^{-1} \nu_{\varphi\, i} \nu_{\varphi\, j} \,, \tag{158}$$

where $\nu_\varphi^{\,i}$ is the unit vector in the direction of the relevant axial Killing vector (60), namely

$$\nu_\varphi^{\,i} = \varpi^{-1} k_3^{\,i} \,, \tag{159}$$

so that its components will be given by $\nu_\varphi^{\,1} = -X^2/\varpi$, $\nu_\varphi^{\,2} = X^1/\varpi$, $\nu_\varphi^{\,3} = 0$. This leads to what is interpretable as the Newtonian limit[11] of a result originally derived in a relativistic context by Bonazzola.[12] In terms of standard cylindrical coordinates, as specified by $X^1 = \varpi \cos\varphi$, $X^2 = \varpi \sin\varphi$, $X^3 = z$, the volume element will be given by $\mathrm{d}X^1\, \mathrm{d}X^2\, \mathrm{d}X^3 = \varpi\, \mathrm{d}\varpi\, \mathrm{d}z\, \mathrm{d}\varphi$. It can thus be seen that the cylindrical homogeneously linear virial theorem obtained by substituting (158) in the generic relation (119) will take the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \rho\, \varpi^2\, \mathrm{d}\varpi\, \mathrm{d}z\, \mathrm{d}\varphi = \int \underset{\rm tot}{T}{}^{ij} \nu_{\varphi\, i} \nu_{\varphi\, j}\, \mathrm{d}\varpi\, \mathrm{d}z\, \mathrm{d}\varphi \,. \tag{160}$$

More particularly, in the stationary axisymmetric case, for which the left hand side vanishes and for which the φ integration is trivial, (160) reduces to the 2-dimensional integral relation

$$\int \underset{\rm tot}{T}{}^{ij} \nu_{\varphi\, i} \nu_{\varphi\, j}\, \mathrm{d}\varpi\, \mathrm{d}z = 0 \,, \tag{161}$$

that was originally derived by Gourgoulhon and Bonazzola,[11] in which it can be seen from (29) that the gravitational contribution to the integrand can be separated out in a decomposition that takes the form

$$\underset{\rm tot}{T}{}^{ij} \nu_{\varphi\, i} \nu_{\varphi\, j} = T_{ij}\, \nu_\varphi^{\,i} \nu_\varphi^{\,j} - \underset{\rm grf}{U} \,, \tag{162}$$

as a consequence of the axisymmetry.

We conclude by emphasising that, like the ordinary virial theorem (153), the Bonazzola theorem (161) is a particular example of the condition

$$\int \underset{\rm tot}{T}{}^{ij} \nabla_i \nabla_j \mathcal{X}\, \mathrm{d}^3 X = 0 \,, \tag{163}$$

that can be seen from (119) to be a generic requirement for any stationary equilibrium configuration for all admissible choices of the weighting factor $\mathcal{X}$. It is to be recalled that T^{ij}_{tot} is the total of all material and gravitational stress contributions, and that, for the purpose of this generic theorem, the relevant admissibility criterion is that $\mathcal{X}$ should be any given function of the Cartesian space coordinates whose gradient components $\nabla_i \mathcal{X}$ increase more slowly than a quadratic function of these coordinates at large distance (this condition would fail for a substitution of the form $\mathcal{X} \mapsto r^3$, but it is evidently satisfied by the substitution $\mathcal{X} \mapsto r^2$ that leads to the ordinary virial theorem, and *a fortiori* by the substitution $\mathcal{X} \mapsto \varpi$ that leads to the Bonazzola virial theorem). Whereas the individual terms in the expansion (152) have forms that depend on the specific nature of the multiconstituent fluid models developed above, on the other hand, the generic relation (163) and its dynamical generalization (119) depend only on the generic form of the energy momentum and mass conservation laws (110), (111) and (113), which should hold for any (complete) Newtonian continuum model. These generally applicable laws are all that is needed for the derivation of the local virial equation (114) that underlies the global relation (119). The generic virial equilibrium relation (163) is just the global consequence of the local equilibrium condition

$$\nabla_i \nabla_j T^{ij}_{\text{tot}} = 0 \tag{164}$$

that obviously holds as the stationary specialization of the local virial equation, namely the dynamical relation (114), whose importance — as an easily memorable law that must be satisfied by any Newtonian continuum configuration — has not been adequately recognized hitherto.

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Appendix. Noether Identities in Newtonian Theory

Let us consider the generic case of a Newtonian model characterized by a total action density that is a sum

$$\Lambda_{\text{tot}} = \sum_a \Lambda_a \tag{A.1}$$

of contributions labelled by some index a. This includes the kind of model dealt with in the preceding work which can be described by taking a to range over the four values specified by the labels "kin", "int", "pot" and "gra". In what follows we shall use the symbol $\cong$ to denote equivalence modulo a spacetime divergence. The content of the action principle is thereby expressible as the postulate that when the relevant field equations are satisfied, any "admissible" infinitesimal variation of the relevant dynamical field variables will give rise to a corresponding total variation

$\delta\Lambda_{\rm tot}$ that is equivalent to zero in this sense, i.e. $\delta\Lambda_{\rm tot} \cong 0$, so that its spacetime integral will vanish by Green's theorem for any variation with compact support.

The purpose of the present section is to consider the effects of more general — not necessarily "admissible" — variations of the relevant dynamical fields as well as variations of the various spacetime background fields on which the complete specification of the action depends. For such more general variations, the total action density variation will satisfy a relation of the form

$$\delta\Lambda_{\rm tot} \cong \sum_a \delta^\ddagger\Lambda_a + \sum_a \delta^\sharp\Lambda_a\,, \tag{A.2}$$

in which $\delta^\ddagger$ denotes the contribution from the variations of the background fields and $\delta^\sharp$ denotes the contribution from any inadmissible parts of the variations of the dynamical fields.

In the technically simpler case of a relativistic model the relevant spacetime background variation would be fully determined by the variation of the spacetime metric $g_{\mu\nu}$, or equivalently of its contravariant inverse $g^{\mu\nu}$. However, in the Newtonian case under consideration here, it is not sufficient to know the variation of the corresponding contravariant space metric $\gamma^{\mu\nu}$. It will be necessary to know as well the associated variation of the corresponding preferred time gradient t_μ. To deal with more elaborate models, the variation of the linear connection might also be involved, but this will not be necessary for models of the simple kind considered here. However to deal with the Galilean gauge dependent contribution $\Lambda_{\rm kin}$ it will also be necessary to take into account the variation of the chosen ether frame vector e^μ. Thus the relevant background field variations will be given by expressions of the form

$$\delta^\ddagger\Lambda_a = \frac{\partial\Lambda_a}{\partial\gamma^{\mu\nu}}\,\delta\gamma^{\mu\nu} + \frac{\partial\Lambda_a}{\partial t_\mu}\,\delta t_\mu + \frac{\partial\Lambda_a}{\delta e^\mu}\,\delta e^\mu\,, \tag{A.3}$$

in which the final term will drop out for gauge independent contributions such as $\Lambda_{\rm mat}$, $\Lambda_{\rm pot}$ and $\Lambda_{\rm grf}$. In the models under consideration, the relevant dynamical fields are the gravitational potential, ϕ, for which arbitrary variations are admissible, and the current vectors $n_{\rm X}^{\,\mu}$ whose admissible variations are restricted, so that a generic variation $\delta n_{\rm X}^{\,\mu}$ may include an inadmissible part $\delta^\sharp n_{\rm X}^{\,\mu}$ that provides a contribution of the form

$$\delta^\sharp\Lambda_a = \frac{\partial\Lambda_a}{\partial n_{\rm X}^{\,\mu}}\,\delta^\sharp n_{\rm X}^{\,\mu}\,. \tag{A.4}$$

As in the (for this purpose simpler) relativistic case, a standard procedure[13] for the derivation of useful Noether type identities is to consider variations of the trivial kind generated by an arbitrary displacement vector field, ξ^μ, say. This means that the variation of each (background or dynamical) field variable will be given by the negative of its Lie derivative. The relevant formulae are thus given by

$$-\delta\Lambda_a = \vec{\xi}\!\!\pounds\,\Lambda_a \equiv \xi^\rho\nabla_\rho\Lambda_a, \tag{A.5}$$

$$-\delta\gamma^{\mu\nu} = \vec{\xi}\!\!\pounds\,\gamma^{\mu\nu} \equiv \xi^\rho\nabla_\rho\gamma^{\mu\nu} - 2\gamma^{\rho\,(\mu}\nabla_\rho\xi^{\nu)}\,, \tag{A.6}$$

$$-\delta t_\mu = \vec{\xi}\!\!\pounds\,t_\mu \equiv \xi^\rho\nabla_\rho t_\mu + t_\rho\nabla_\mu\xi^\rho\,, \tag{A.7}$$

$$-\delta e^\mu = \vec{\xi}\!\!\pounds\,e^\mu \equiv \xi^\rho\nabla_\rho e^\mu - e^\rho\nabla_\rho\xi^\mu \tag{A.8}$$

and

$$-\delta n_{\rm X}^{\,\mu} = \vec{\xi}\!\!\pounds\,n_{\rm X}^{\,\mu} \equiv \xi^\rho\nabla_\rho n_{\rm X}^{\,\mu} - n_{\rm X}^{\,\rho}\nabla_\rho\xi^\mu\,. \tag{A.9}$$

In view of the uniformity properties (I-16) and (I-18) of the unperturbed background fields, we shall be left with

$$\delta\gamma^{\mu\nu} = 2\gamma^{\rho\,(\mu}\nabla_\rho\xi^{\nu)}\,, \tag{A.10}$$

$$\delta t_\mu = -t_\rho\nabla_\mu\xi^\rho\,, \tag{A.11}$$

$$\delta e^\mu = e^\rho\nabla_\rho\xi^\mu\,, \tag{A.12}$$

and it can be seen from the expression (I-86) for the form of an admissible current variation that the only remaining (i.e. inadmissible) contribution from (A.9) will be given simply by

$$\delta^\sharp n_{\rm X}^{\,\mu} = n_{\rm X}^{\,\mu}\nabla_\rho\xi^\rho\,. \tag{A.13}$$

Since (using the notation introduced in (A.2) for equivalence modulo a divergence) the variation (A.5) will evidently satisfy

$$\delta\Lambda_a \cong \Lambda_a\nabla_\rho\xi^\rho\,, \tag{A.14}$$

it can be seen that the terms in (A.2) can be regrouped on the left to give a relation of the simple form

$$T_{{\rm tot}\ \nu}^{\ \ \mu}\nabla_\mu\xi^\nu \cong 0\,, \tag{A.15}$$

in which the quantity that will be interpretable as the relevant total stress momentum energy density tensor can be read out in the form

$$T_{{\rm tot}\ \nu}^{\ \ \mu} = \sum_a T_{a\,\nu}^{\,\mu}\,, \tag{A.16}$$

as the sum of contributions given by the formula

$$T_{a\,\nu}^{\,\mu} = \Psi_a\delta_\nu^\mu - 2\frac{\partial\Lambda_a}{\partial\gamma^{\rho\nu}}\,\gamma^{\rho\mu} + \frac{\partial\Lambda_a}{\partial t_\mu}\,t_\nu - \frac{\partial\Lambda_a}{\partial e^\nu}\,e^\mu\,, \tag{A.17}$$

in which the relevant generalized pressure contribution Ψ_a is given in terms of the corresponding momentum contributions,

$$\pi_{a\ \mu}^{\ {\rm X}} = \frac{\partial\Lambda_a}{\partial n_{\rm X}^{\,\mu}}\,, \tag{A.18}$$

by

$$\Psi_a = \Lambda_a - \sum \pi_{a\ \mu}^{\ {\rm X}} n_{\rm X}^{\,\mu}\,. \tag{A.19}$$

It is to be remarked that due to the restrictions

$$t_\nu\,\delta\gamma^{\nu\mu} = -\gamma^{\mu\nu}\,\delta t_\nu\,, \quad t_\nu\,\delta e^\nu = -e^\nu\,\delta t_\nu\,, \tag{A.20}$$

resulting from (I-3) and (I-16), there is some gauge ambiguity in the specification of the partial derivative coefficients introduced in (A.3) but it can easily be checked that the stress momentum energy contributions $T_{a\,\nu}^{\,\mu}$ specified by the combination (A.17) will be physically well-defined in the sense of being unaffected by the choice adopted in (A.2).

By a further equivalence transformation (modulo a divergence) it can be seen that (A.15) can be converted to the form

$$\xi^\nu \nabla_\mu T_{\!\!\text{tot}\,\nu}^{\;\;\mu} \cong 0\,. \tag{A.21}$$

Since this must hold for an arbitrary vector field ξ^μ, and hence in particular for a displacement field with compact support in any small spacetime neighborhood, it can be seen by integrating over such a neighborhood (so that the divergence ambiguity in the equivalence relation cancels out by Green's theorem, as in the usual derivation of the dynamical equations from the action principle) that the coefficient of ξ^μ in (A.21) must vanish, i.e. we obtain a total energy momentum conservation law of the form

$$\nabla_\mu T_{\!\!\text{tot}\,\nu}^{\;\;\mu} = 0\,. \tag{A.22}$$

It remains to show that the conserved total stress momentum energy density tensor $T_{\!\!\text{tot}\,\nu}^{\;\;\mu}$ obtained by the foregoing procedure is actually the same as the tensor $T_{\!\!\text{tot}\,\nu}^{\;\;\mu}$ introduced in (30)

The relation (A.22) is not an identity in the strict sense, since its derivation depends on the dynamical field equations obtained from the action principle. To obtain a purely mathematical identity it is necessary to take account of the unrestrained variations of all relevant field variables. In the application with which we are concerned here this includes not just the background field variations involved in (A.3) and the variations of the currents $n_{\rm x}^{\mu}$ but also the variations of the gravitational potential ϕ and its gradient, so that the complete generic expression for the variation of an action density contribution Λ_a will have the form

$$\delta\Lambda_a = \delta^\ddagger\Lambda_a + \sum \pi_{a\;\mu}^{\;\rm x}\,\delta n_{\rm x}^{\mu} + \frac{\partial\Lambda_a}{\partial\phi}\delta\phi + \frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\,\delta(\phi_{,\mu})\,. \tag{A.23}$$

For variations generated, as before, by an arbitrary displacement vector field ξ^μ, we can again use the background field variation formulae (A.10), (A.11) and (A.12), together with the corresponding dynamical field variation formulae (A.9) and

$$\delta\phi = -\vec{\xi}\pounds\,\phi = -\xi^\mu\phi_{,\mu}\,, \tag{A.24}$$

we thus obtain a relation of the form

$$\xi^\nu\left(\nabla_\nu\Lambda_a - \sum_{\rm x}\pi^{\rm x}_{a\,\mu}\,\nabla_\nu n^\mu_{\rm x} - \frac{\partial\Lambda_a}{\partial\phi}\nabla_\nu\phi - \frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\,\nabla_\mu\nabla_\nu\phi\right)$$

$$= \left(\frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\,\nabla_\nu\phi - \sum_{\rm x}\pi^{\rm x}_{a\,\nu}\,n^\mu_{\rm x} - 2\frac{\partial\Lambda_a}{\partial\gamma^{\rho\nu}}\,\gamma^{\rho\mu} + \frac{\partial\Lambda_a}{\partial t_\mu}\,t_\nu - \frac{\partial\Lambda_a}{\partial e^\nu}\,e^\mu\right)\nabla_\mu\xi^\nu\,, \tag{A.25}$$

which will be satisfied as an identity in the sense of being independent of the dynamical field equations. Since it is possible to choose both ξ^ν and its derivative $\nabla_\mu\xi^\nu$ independently at any given point, it follows that the corresponding coefficients must vanish. Thus from the left hand side of (A.25) we obtain the obvious identity

$$\nabla_\nu\Lambda_a = \sum\pi^{\rm x}_{a\,\mu}\nabla_\nu n^\mu_{\rm x} + \frac{\partial\Lambda_a}{\partial\phi}\nabla_\nu\phi + \frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\,\nabla_\mu\nabla_\nu\phi\,, \tag{A.26}$$

while from the right hand side we obtain an identity of the less trivial form

$$\frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\,\nabla_\nu\phi - \sum\pi^{\rm x}_{a\,\nu}\,n^\mu_{\rm x} - 2\frac{\partial\Lambda_a}{\partial\gamma^{\rho\nu}}\,\gamma^{\rho\mu} + \frac{\partial\Lambda_a}{\partial t_\mu}\,t_\nu - \frac{\partial\Lambda_a}{\partial e^\nu}\,e^\mu = 0\,. \tag{A.27}$$

A noteworthy special case is that of the internal contribution $\Lambda_{\rm int}$, which depends neither on the gravitational potential ϕ (unlike $\Lambda_{\rm pot}$) nor on the ether frame vector e^μ (unlike $\Lambda_{\rm kin}$), so that when it is substituted for Λ_a in (A.27) the first and the last term will both drop out. Contraction with t_μ and $\gamma^{\nu\sigma}$ can then be used to eliminate the other terms at the end, leaving a result that can be recognized as the simple identity (I-143) that was quoted above, while the following identity (I-144) can similarly be derived by performing an antisymmetrization after contracting with just $\gamma^{\nu\sigma}$.

Quite generally, the Noether identity (A.27) can be used to replace the formula (A.17) by the more explicit and manifestly gauge independent expression

$$T^{\ \mu}_{a\,\nu} = \Psi_a\delta^\mu_\nu + \sum\pi^{\rm x}_{a\,\nu}\,n^\mu_{\rm x} - \frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\,\nabla_\nu\phi\,. \tag{A.28}$$

With the aid of (A.26) it can be directly verified that each such contribution will satisfy a divergence identity of the form

$$\nabla_\mu T^{\ \mu}_{a\,\nu} = \sum f^{\rm x}_{a\,\nu} + \frac{\delta\Lambda_a}{\delta\phi}\,\nabla_\nu\phi\,, \tag{A.29}$$

in which the Eulerian derivative has a definition of the usual form

$$\frac{\delta\Lambda_a}{\delta\phi} = \frac{\partial\Lambda_a}{\partial\phi} - \nabla_\mu\left(\frac{\partial\Lambda_a}{\partial(\phi_{,\mu})}\right), \tag{A.30}$$

while the force density contributions are given by an expression whose form

$$f^{\rm x}_{a\,\mu} = 2n^{\ \nu}_{\rm x}\nabla_{[\nu}\pi^{\rm x}_{a\,\mu]} + \pi^{\rm x}_{a\,\mu}\nabla_\nu n^{\ \nu}_{\rm x} \tag{A.31}$$

is analogous to that for the combined force densities (I-159).

By summing over these separate contributions we obtain the corresponding formula

$$T_{\text{tot}}{}^{\mu}{}_{\nu} = \Psi_{\text{tot}}\delta^{\mu}_{\nu} + \sum_{\text{x}} \pi_{\text{tot}}{}^{\text{x}}{}_{\nu}\, n^{\mu}_{\text{x}} - \frac{\partial\Lambda_{\text{tot}}}{\partial(\phi_{,\mu})}\nabla_{\nu}\phi\,, \tag{A.32}$$

for the total stress momentum energy density tensor, where the corresponding total pressure function and momenta are given by

$$\Psi_{\text{tot}} = \sum_a \Psi_a\,, \qquad \pi_{\text{tot}}{}^{\text{x}}{}_{\nu} = \sum_a \pi_a{}^{\text{x}}{}_{\nu}\,, \tag{A.33}$$

while there will be an analogous expression

$$f_{\text{tot}}{}^{\text{x}}{}_{\nu} = \sum_a f_a{}^{\text{x}}{}_{\nu}\,, \tag{A.34}$$

for the total non-gravitational force density exerted on each constituent.

In the application considered in the preceding work the purely gravitational action density Λ_{grf} given by (25) provides no contribution to the momenta, i.e. we shall have $\pi_{\text{grf}}{}^{\text{x}}{}_{\nu} = 0$, and hence $\Psi_{\text{grf}} = \Lambda_{\text{grf}}$ so we can make the identifications $\pi_{\text{tot}}{}^{\text{x}}{}_{\nu} = \pi^{\text{x}}_{\nu}$, $f_{\text{tot}}{}^{\text{x}}{}_{\nu} = f^{\text{x}}_{\nu}$ and $\Psi_{\text{tot}} = \Psi + \Lambda_{\text{grf}}$ in which it is to be recalled that we use a blank label to indicate the sum over all values of a except the gravitational contribution "gra", i.e. for the sum over the values "kin", "int" and "pot", which is the only part that is relevant if (as in the Cowling approximation) one is concerned just with evolution in a fixed gravitational background, but not with the effects of self gravity. It can thus be verified that the definitions introduced in the systematic mathematical procedure developed in this appendix are entirely consistent with those introduced *ad hoc* in the main part of the text. In particular, it can easily be seen from (A.32) that the general purpose prescription provided by (A.17) and (A.18) for $T_{\text{tot}}{}^{\mu}{}_{\nu}$ leads to a result that is in exact agreement with what is given by the formula (30) that was obtained from more specific physical considerations in the main part of the text, while similarly by summing over the index a in the identity (A.31) one recovers a result that can be seen to agree with the previously quoted divergence formula (3).

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