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RESEARCH ARTICLE

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Covariant Newtonian and relativistic dynamics of (magneto)-elastic solid model for neutron star crust

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Abstract This work develops the dynamics of a perfectly elastic solid model for application to the outer crust of a magnetised neutron star. Particular attention is given to the Noether identities responsible for energy-momentum conservation, using a formulation that is fully covariant, not only (as is usual) in a fully relativistic treatment but also (sacrificing accuracy and elegance for economy of degrees of gravitational freedom) in the technically more complicated case of the Newtonian limit. The results are used to obtain explicit (relativistic and Newtonian) formulae for the propagation speeds of generalised (Alfven type) magneto-elastic perturbation modes.

Keywords Newtonian limit · Elastic solid · Noether identity

1 Introduction

In astrophysical contexts of the kind exemplified by a neutron star crust, it is commonly necessary to go back and forth between relativistic models having the advantage of greater elegance and in principle – particularly at a global level – of higher accuracy, and Newtonian models that are more convenient from the point of view of other considerations such as economy in gravitational degrees of freedom, and availability of detailed underlying descriptions at a microscopic level. As a consequence of the fact that – unlike the Galilean invariance group – the Lorentz group is semi-simple, there are contexts (e.g. involving superfluidity [1, 2]) in which a fully relativistic treatment is actually easier to implement than a corresponding Newtonian treatment, even though the latter would be perfectly

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adequate as far as accuracy is concerned. On the other hand there are many contexts – particularly those involving electromagnetic effects or strong gravitational fields – in which a relativistic treatment would be indispensible if high accuracy were required, but for which a Newtonian treatment might nevertheless be easier to implement as a first approximation.

To facilitate transition between these two complementary kinds of description, it is desirable to develop a unified treatment in which both relativistic and Newtonian models are described in terms of technical machinery and terminology that are compatible as much as possible, so as to be consistent in the limit when the relativistic spacetime metric goes over to the degenerate spacetime structure of Newtonian theory. In a coherent approach of this kind, the Newtonian limit will naturally be obtained in fully covariant formulation of the kind [3–5] whose mathematical machinery was first developed by Cartan [6]. In a preceding series of articles [7–9] on multiconstituent fluid and superfluid dynamics, it was shown how such a 4 dimensionally covariant formulation of Newtonian theory can provide physical insights (e.g concerning concepts such as helicity) that are not so easy to obtain by the traditional approach based on a 3+1 space time decomposition.

Continuing in the same spirit, the purpose of the present work is to contribute to the further development of the unified treatment of relativistic and Newtonian theory by treating the case of elastic solid models, of the kind appropriate for the description of the outer crust of a neutron star, including the magneto-elastic case (that arises in the limit of perfect electrical conductivity) for which the elastic structure is modified by a frozen-in magnetic field, of the kind whose effects are observed in pulsars. (The category of such models includes the special case of ordinary – fluid not solid – perfect magnetohydrodynamics in the limit of negligible elastic rigidity.)

Accurate description of such stars at a global level (not to mention a recently proposed cosmological application [10]) requires a general relativistic treatment, but use of a flat space background will be sufficient for treatment of the local mechanical properties to be considered here. In ordinary pulsars the magnetic field is sufficiently low that (except in the outer skin and the magnetosphere outside, where the matter density is comparatively low) such a flat background space treatment can be carried out (as described below) within a purely Newtonian framework. However a fully (at least special, if not general) relativistic treatment will be indispensible even locally (in a Minkowski background) when the magnetic field is sufficiently strong, as will be the case not just near the surface, but even in the deeper layers, for the special class of pulsars known as magnetars. The relativistic version of the magneto-elastic treatment developed here is particularly relevant for such strongly magnetised ($B \gtrsim 10^{14} G$) neutron stars, in which flares powered by magnetic stress are believed to be responsible for gamma ray bursts of the brief but intense kind observed in soft gamma repeaters, the most spectacular example so far – the most intense ever observed in our galaxy – having been the 27 December event that occurred in SGR $1806-20$ in 2004 [11]. For a complete treatment of such a flare a fully general relativistic description would presumably be necessary since a phenomenon of this kind is though to be attributable to a global modification of the magnetic field of the neutron star [12, 13].

As an application, in both the Newtonian and fully relativistic cases such that the underlying solid is in a simple isotropic state, the relevant (rigidity modified) propagation speeds of (Alfven type) perturbation modes are explicitly evaluated.

A subsequent article will be needed to treat the more elaborate kind of model needed for the innermost layers of a neutron star crust, in which an ionic solid lattice is permeated by an independently moving current of superfluid neutrons.

2 Milne structure of Newtonian spacetime

Before proceeding, let us recapitulate the geometric essentials of Newtonian space time structure in a 4-dimensional background with respect to an arbitrary system of local coordinates x^μ, $\mu = 0, 1, 2, 3$, as described in greater detail in the first article [7] of the preceding series.

The specification of a relativistic structure is fully determined by a non degenerate spacetime metric tensor having components $g_{\mu\nu}$, which in the special relativistic case are required to be constant in a prefered class of Minkowski type coordinate systems. This tensor will have a well defined contravariant inverse $g^{\mu\nu}$, from which a corresponding connection (which will vanish with respect to Minkowski coordinates in special relativity) with components $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ that are unambiguously obtainable using the usual Riemannian formula $\Gamma_{\mu}{}^{\nu}{}_{\rho} = g^{\nu\sigma}(g_{\sigma(\mu,\rho)} - {}^1/_2\, g_{\mu\rho,\sigma})$ using a comma to denote partial differentiation and round brackets to indicate index symmetrisation.

Newtonian theory is traditionally formulated in terms of an Aristotelian frame, meaning a direct product of a 1-dimensional trajectory parametrised by the Newtonian time t and a flat 3-dimensional Euclidean space whose positive definite metric gives rise to a corresponding 4-dimensional metric $\eta_{\mu\nu}$ that is of degenerate, rank-3, positive indefinite type, so that it does not determine a contravariant inverse tensor, and that, unlike its relativistic analogue, is not physically well defined because it depends on the choice of the Aristotelian "ether" frame, as characterised by the choice of a unit ether flow vector, e^μ say, that will be a null eigenvector of the corresponding degenerate metric, i.e. that will satisfy $\eta_{\mu\nu}e^\nu = 0$. There is however a complementary tensorial "Coriolis" structure that (unlike the Aristotelian structure) is physically well defined in the sense of being preserved by the allowable (time foliation preserving) ether gauge transformations. This invariant structure consists of the time gradient 1-form $t_\mu = t_{,\mu}$ and a contravariant metric tensor $\gamma^{\mu\nu}$ that (like its gauge dependent covariant complement $\eta_{\mu\nu}$)has the property of being degenerate, of rank-3 positive indefinite type, with null eigendirection determined by the time covector, i.e.

$$\gamma^{\mu\nu}t_\nu = 0\,. \tag{1}$$

Although (like the non-degenerate metric in the relativistic case) this associated pair of tensors t_μ and $\gamma^{\mu\nu}$ is physically well defined, nevertheless the specification of this "Coriolis" structure (unlike that of the non-degenerate metric in the relativistic case) is not by itself sufficient to fully determine the geometric structure of spacetime in the Newtonian limit, and furthermore (like the metric in special relativity but not in general relativity) it is not freely variable over spacetime.

To start with, the Coriolis structure specified by the fields $\gamma^{\mu\nu}$ and t_μ is required to be *flat* in the sense that there should exist coordinates with respect to which the corresponding field components are constant. It follows more particularly that there will be coordinates of a preferred type with respect to which these fields will have components of standard Aristotelian-Cartesian form, as given in terms of Kronecker notation by $\gamma^{\mu\nu} = \delta^{\mu}_{1}\delta^{\nu}_{1} + \delta^{\mu}_{2}\delta^{\nu}_{2} + \delta^{\mu}_{3}\delta^{\nu}_{3}$, and $t_\mu = \delta^0_\mu$. Such a coordinate system will determine a corresponding Aristotelian "ether" frame vector with components $e^\mu = \delta^{\mu}_{0}$ and its associated covariant space metric with components $\eta_{\mu\nu} = \delta^1_\mu\delta^1_\nu + \delta^2_\mu\delta^2_\nu + \delta^3_\mu\delta^3_\nu$, and $t_\mu = \delta^0_\mu$. (In the preceding work [7] this quantity $\eta_{\mu\nu}$ was written as $\gamma_{\mu\nu}$, using the same Greek letter gamma as for its contravariant analogue, but in the present work the symbol γ will be reserved for quantities that are gauge independent.) The Aristotelian-Cartesian coordinate system will also determine a corresponding symmetric connection, namely the one whose components

$$\Gamma_{\mu\ \rho}^{\ \nu} = \Gamma_{\rho\ \mu}^{\ \nu} \tag{2}$$

will vanish in that system, a requirement which evidently ensures that it will have vanishing curvature. The corresponding covariant differentiation operator ∇ will satisfy the commutation relation

$$\nabla_\rho \nabla_\sigma - \nabla_\sigma \nabla_\rho = 0\,, \tag{3}$$

and will be such that the corresponding covariant derivatives of the tensor fields $\gamma^{\mu\nu}$ and t_μ will vanish:

$$\nabla_\rho \gamma^{\mu\nu} = 0 \quad \nabla_\rho t_\mu = 0\,. \tag{4}$$

As well as satisfying the algebraic conditions

$$e^\mu t_\mu = 1 \quad e^\mu \eta_{\mu\nu} = 0, \quad \eta_{\mu\rho}\gamma^{\rho\nu} = \eta^\nu_\mu = \delta^\nu_\mu - e^\nu t_\mu\,, \tag{5}$$

the ether frame dependent fields e^μ , $\eta_{\mu\nu}$, and the associated space projection tensor η^ν_μ introduced in (5), will also have corresponding covariant derivatives that vanish:

$$\nabla_\rho e^\mu = 0, \quad \nabla_\rho \eta_{\mu\nu} = 0 \quad \nabla_\rho \eta^\nu_\mu = 0\,. \tag{6}$$

These fields can be used to specify an ether frame dependent Lorentz signature metric defined by $\overline{g}_{\mu\nu} = -t_\mu t_\nu + \eta_{\mu\nu}$, with contravariant inverse, given by $\underline{g}^{\mu\nu} = -e^\mu e^\nu + \gamma^{\mu\nu}$, whose determinant provides a 4-dimensional volume measure that (modulo a sign ambiguity depending on a choice of parity orientation) fixes a corresponding antisymmetric tensor with components $\overline{\varepsilon}^{\mu\nu\rho\sigma} = \overline{\varepsilon}^{[\mu\nu\rho\sigma]}$ (using square brackets to denote index antisymmetrisation). This measure tensor is alternatively definable directly by the condition that its components with respect to Aristotelian-Cartesian coordinates (with the chosen orientation) should be given by $+1$ or -1 whenever the indices are respectively even or odd permutations of the sequence $\{0, 1, 2, 3\}$. The corresponding antisymmetric covariant measure 4-form $\underline{\varepsilon}_{\mu\nu\rho\sigma} = \underline{\varepsilon}_{[\mu\nu\rho\sigma]}$ is then specifiable (in the manner that is familiar in the context of relativistic theory) by the normalisation condition

$$\overline{\varepsilon}^{\mu\nu\rho\sigma}\underline{\varepsilon}_{\mu\nu\rho\sigma} = -4!\,. \tag{7}$$

It can be seen that (unlike e^μ and $\eta_{\mu\nu}$ and the frame dependent Lorentz metric) they share with t_μ and $\gamma^{\mu\nu}$ the property of being independent of the choice of ether gauge. These tensors will evidently give rise to purely spacelike 3 index analogues, defined as the Hodge type duals of t_μ and e^μ respectively, namely the gauge independent space alternating tensor

$$\epsilon^{\mu\nu\rho} = \overline{\varepsilon}^{\mu\nu\rho\sigma} t_\sigma \,, \tag{8}$$

and the frame dependent 3-form

$$e^*_{\mu\nu\rho} = e^\lambda \underline{\varepsilon}_{\lambda\mu\nu\rho} \tag{9}$$

which is interpretable as representing an ether current. It is evident that all these tensors will share the flatness property characterised by the connection, in the sense of satisfying the conditions

$$\nabla_\lambda \overline{\varepsilon}^{\mu\nu\rho\sigma} = 0, \quad \nabla_\lambda \underline{\varepsilon}_{\mu\nu\rho\sigma} = 0\,. \tag{10}$$

and

$$\nabla_\lambda \epsilon^{\mu\nu\rho} = 0, \quad \nabla_\lambda e^*_{\mu\nu\rho} = 0\,. \tag{11}$$

Despite the simplification provided by the flatness property that is expressed by (4), (6), (10) and (11), the Newtonian case is subject to the complication that neither the flat coordinate system nor even the corresponding flat connection is unambiguously determined by the tensor fields $\gamma^{\mu\nu}$ and t_μ. The standard form expressible by $\gamma^{\mu\nu} = \delta^\mu_{\underset{1}{}}\delta^\nu_{\underset{1}{}} + \delta^\mu_{\underset{2}{}}\delta^\nu_{\underset{2}{}} + \delta^\mu_{\underset{3}{}}\delta^\nu_{\underset{3}{}}$ and $t_\mu = \delta^0_\mu$ will in fact be preserved by a large category of transformations that is known as the Coriolis group, which includes not only boosts but also time dependent rotations. However the physical structure of Newtonian spacetime is not preserved by time dependent rotations, but only by transformations of a more restricted category known as the Milne group, which is characterised by the condition that the modification of the ether frame vector should depend only on the Newtonian time t, as specified modulo a choice of origin by $t_{,\mu} = t_\mu$. This means that the transformation of the ether vector will be expressible in the form

$$e^\nu \mapsto \breve{e}^\mu = e^\mu + b^\mu \,, \tag{12}$$

for a boost vector field b^μ that is subject to the condition

$$b^\mu t_\mu = 0\,, \quad \gamma^{\nu\rho}\nabla_\rho b^\mu = 0 \tag{13}$$

and for which the corresponding acceleration vector will be specified by

$$a^\mu = e^\rho \nabla_\rho b^\mu\,, \quad a^\mu t_\mu = 0\,. \tag{14}$$

The ensuing transformation of the covariant space metric will be given by

$$\eta_{\mu\nu} \mapsto \breve{\eta}_{\mu\nu} = \eta_{\mu\nu} - 2t_{(\mu}\eta_{\nu)\rho} b^\rho + \eta_{\rho\sigma} b^\rho b^\sigma t_\mu t_\nu \,, \tag{15}$$

while those of the corresponding space projection and space measure tensors will be

$$\eta^\nu_\mu \mapsto \eta^\nu_\mu - b^\nu t_\mu\,, \quad e^*_{\mu\nu\rho} \mapsto \breve{e}^*_{\mu\nu\rho} = e^*_{\mu\nu\rho} + b^\lambda \underline{\varepsilon}_{\lambda\mu\nu\rho}\,. \tag{16}$$

Unlike the ether vector e^μ, the covariant space metric $\eta_{\mu\nu}$, and the frame dependent Lorentz signature metric $g_{\mu\nu}$ that was invoked above, the tensors constituting the Coriolis structure, namely the time gradient covector t_μ, the contravariant space metric $\gamma^{\mu\nu}$, and also the associated space-time measure given by $\underline{\varepsilon}_{\mu\nu\rho\sigma}$ or $\overline{\varepsilon}^{\mu\nu\rho\sigma}$ as well as the corresponding spacelike alternating tensor $\epsilon^{\mu\nu\rho}$ are all physically well defined in the sense of being independent of the choice of Aristotelian frame, since the effect on them of the boost transformation specified by (12) will be given trivially by

$$\breve{t}_\mu = t_\mu \qquad \breve{\gamma}^{\mu\nu} = \gamma^{\mu\nu} \,, \tag{17}$$

and

$$\breve{\underline{\varepsilon}}_{\mu\nu\rho\sigma} = \underline{\varepsilon}_{\mu\nu\rho\sigma} \,, \quad \breve{\overline{\varepsilon}}^{\mu\nu\rho\sigma} = \overline{\varepsilon}^{\mu\nu\rho\sigma} \,, \quad \breve{\epsilon}^{\mu\nu\rho} = \epsilon^{\mu\nu\rho} \,. \tag{18}$$

Within the full group constituted by the (in general non linear) Milne transformations characterised by (12) and (13), there is a linear subclass constituting the well known Galilei group, that is characterised by the requirement of preservation of the connection and the associated covariant differentiation operator ∇, for which the necessary and sufficient condition is that the boost acceleration vector a^μ should vanish. However for a generic Milne transformation the covariant differentiation operator will undergo a non trivial transformation, $\nabla \mapsto \breve{\nabla}$, specified by a corresponding transformation of the connection that will be given, using the definition (14), by the formula

$$\Gamma_{\mu\ \rho}^{\ \nu} \mapsto \breve{\Gamma}_{\mu\ \rho}^{\ \nu} = \Gamma_{\mu\ \rho}^{\ \nu} - t_\mu a^\nu t_\rho \,, \quad , \tag{19}$$

which has the noteworthy property of preserving the trace of the connection, i.e. we shall have

$$\breve{\Gamma}_{\mu\ \rho}^{\ \mu} = \Gamma_{\mu\ \rho}^{\ \mu} \,, \tag{20}$$

with the implication that for the evaluation of the simple divergence of a contravariant vector field, such as the displacement field ξ^μ introduced below, it will not matter which connection we use, i.e. we shall have $\breve{\nabla}_\mu \xi^\mu = \nabla_\mu \xi^\mu$

Instead of working with the kind of flat but ether gauge dependent connection that has just been described, it is useful for some purposes to work instead with a curved but gauge independent connection of a gravitational field dependent kind, that was first introduced by Cartan, and that is described in the preceding work referred to above [7]. However such a Cartan connection will not be needed in the present work.

3 Relativistic correspondence

A Newtonian space time structure of the kind described in the preceding section can be obtained as a low velocity limit from a corresponding relativistic theory on the supposition that the latter is approximately describable, in terms of an adjustable parameter c, by a Lorentz signature metric $\mathrm{d}\tilde{s}^2 = \tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, having the form

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} - \tilde{c}^2\, t_\mu\, t_\nu \,, \tag{21}$$

while, according to the preceding relations (5), the corresponding contravariant metric will be given by

$$\tilde{g}^{\mu\nu} = \gamma^{\mu\nu} - \frac{1}{\tilde{c}^2}\, e^\mu\, e^\nu \,. \tag{22}$$

The quantity $\tilde{c}$ in these expressions is interpretable as representing the speed of light with respect to coordinates of the standard Aristotelian kind as described in the preceding section. In the absence of gravitational effects the metric (21) can be taken to be of flat Minkowski type, as given by a fixed value for $\tilde{c}$, but to allow for the effect of a Newtonian gravitational potential, ϕ say, it is necessary to take it to be given in terms of a fixed asymptotic value, c, by the formula

$$\tilde{c}^2 = c^2 + 2\phi \, . \tag{23}$$

The degenerate Newtonian structure of the preceding section is then obtained by taking the limit $c \to \infty$, which evidently gives

$$\tilde{g}^{\mu\nu} \to \gamma^{\mu\nu} \, , \quad c^{-2}\tilde{g}_{\mu\nu} \to -t_\mu t_\nu \, . \tag{24}$$

It is to be remarked that although the spacetime metric itself is degenerate in this Newtonian limit, the associated Riemannian connection,

$$\tilde{\Gamma}_\mu{}^\rho{}_\nu = \tilde{g}^{\rho\sigma}(\tilde{g}_{\sigma(\mu,\nu)} - \tfrac{1}{2}\tilde{g}_{\mu\nu,\sigma}) \, , \tag{25}$$

will be well behaved, and that it will agree with the flat connection $\Gamma_\mu{}^\rho{}_\nu$ of the preceding section in the absence of a gravitational field, i.e. when the potential ϕ in (23) is uniform. There will however be a difference in the generic case, for which it can be seen that the large c limit will be given by the relation

$$\tilde{\Gamma}_\mu{}^\rho{}_\nu \to \Gamma_\mu{}^\rho{}_\nu + t_\mu t_\nu \gamma^{\rho\sigma}\phi_{,\sigma} \, . \tag{26}$$

This shows that in the Newtonian limit $\tilde{\Gamma}_\mu{}^\rho{}_\nu$ goes over to the Newton-Cartan connection that was denoted by $\omega_\mu{}^\rho{}_\nu$ in the preceding work [7]. This means that the associated Riemannian covariant differentiation operator $\tilde{\nabla}_\nu$ will go over, not to the usual flat space differentiation operator ∇_ν of the preceding section, but to the Cartan type differentiation operator that was denoted in the preceding work [7] by D_ν.

In Newtonian dynamical theory, the concept of mass conservation plays an essential role. In relativistic theory, mass as such is not in general conserved, but in relevant applications it will nevertheless be possible to attribute most of the mass to other effectively conserved currents (e.g. that of baryons in a typical astrophysical context, or those of separate chemical elements in a typical non-nuclear terrestrial context). Such conserved currents can be endowed with suitable mass weighting factors – e.g. the rest mass, m say, of an isolated proton or of a neutral hydrogen atom in the baryonic case – so as to provide what is needed in a Newtonian limit. A conserved current can be represented – without reference to any spacetime metric structure – as a Cartan type 3-form, with antisymmetric components $n^*_{\mu\nu\rho}$ say, that is closed in the sense of having vanishing exterior derivative. This closure condition will be equivalent to that of vanishing of the divergence of the corresponding current 4-vector that is given by the (Hodge type duality) ansatz

$$n^\mu = \frac{c}{3!}\tilde{\varepsilon}^{\mu\nu\rho\sigma}\, n^*_{\nu\rho\sigma} \, , \tag{27}$$

in terms of the antisymmetric measure tensor associated with the spacetime metric $\tilde{g}_{\mu\nu}$, as given in terms of the modulus of the metric determinant $|\tilde{g}|$ by the condition that its components should be given by $+\|\tilde{g}\|^{1/2}$ or $-\|\tilde{g}\|^{1/2}$ whenever the indices are respectively even or odd permutations of the sequence {0, 1, 2, 3}. This means that it will be related to the non-relativistic spacetime measure of the preceding section by $\tilde{\varepsilon}_{\mu\nu\rho\sigma} = \tilde{c}\,\underline{\varepsilon}_{\mu\nu\rho\sigma}$ and $\tilde{\varepsilon}^{\mu\nu\rho\sigma} = \tilde{c}^{-1}\overline{\varepsilon}^{\mu\nu\rho\sigma}$, so that in the limit $c \to \infty$ one obtains $n^{*}_{\mu\nu\rho} \mapsto \underline{\varepsilon}_{\mu\nu\rho\sigma} n^{\sigma}$).

A multiconstituent system may involve several constituent currents $n^{\nu}_{\rm X}$, which need not be separately conserved, but that combine to give a locally conserved total

$$n^{\nu} = \Sigma_{\rm X} n^{\nu}_{\rm X}\,, \quad \tilde{\nabla}_{\nu} n^{\nu} = 0\,. \tag{28}$$

For a confined system, the corresponding globally conserved mass integral, M say, associated with a spacelike hypersurface – as specified by a fixed value of some suitable time coordinate x^0 – will be given in terms of the other coordinates and of the relevant mass parameter m by

$$M = m\int n^{*}_{123}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\,, \quad n^{*}_{123} = c^{-1}\tilde{\varepsilon}_{1230} n^0\,. \tag{29}$$

In the applications under consideration, the evolution equations of the relevant currents will be obtainable from a relativistic action principle of the world-line variational kind that is indispensible [15] for treatment of a constituent that is solid, and that is very suitable [1, 2, 16] (though other – e.g. Clebsch type [17] – possibilities exist) for treatment of a medium in which the relevant constituents are fluid. This kind of variational principle is based on a relativistic action integral

$$\tilde{\mathcal{I}} = \int \tilde{\Lambda}\, c^{-1}\tilde{\varepsilon}_{1230}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\,\mathrm{d}x^0\,, \tag{30}$$

for which the action density will be decomposible in the form

$$\tilde{\Lambda} = \Lambda_{\rm bal} + \Lambda_{\rm int}\,, \tag{31}$$

in which $\Lambda_{\rm int}$ is a relatively small intrinsic contribution. The dominant extrinsic contribution $\Lambda_{\rm bal}$ is the ballistic part – which is all that would be needed for the case of force free geodesic motion – namely the negative of the sum of the relevant rest mass-energy contributions, as given by

$$\Lambda_{\rm bal} = -m\,c^2\,\Sigma_{\rm X} n_{\rm X}\,, \tag{32}$$

where m is the appropriate mass weighting factor and, for each constituent, $n_{\rm X}$ is the corresponding number density as evaluated in the relevant rest frame. The frame in question is characterised by the corresponding timelike frame vector $u^{\nu}_{\rm X}$ that is specified, subject to the normalisation conditions $u^{\nu}_{\rm X} u_{{\rm X}\nu} = -c^2$, by the expressions

$$n^{\nu}_{\rm X} = n_{\rm X}\,u^{\nu}_{\rm X}\,, \quad n_{\rm X} = c^{-1}\left(-n^{\nu}_{\rm X} n_{{\rm X}\nu}\right)^{1/2}\,. \tag{33}$$

To obtain the Newtonian limit in which we are ultimately interested, with respect to the gauge specified by some chosen ether vector e^{μ}, it is convenient to express each frame vector $u^{\nu}_{\rm X}$ in terms of a corresponding purely spacelike 3-velocity vector $v^{\nu}_{\rm X}$ in the form

$$u^{\nu}_{\rm X} = u^{0}_{\rm X}\left(e^{\nu} + v^{\nu}_{\rm X}\right)\,, \quad v^{\nu}_{\rm X}\,t_{\nu} = 0\,, \tag{34}$$

in which it can be seen that the required normalisation factor is identifiable as the the time component of the frame vector with respect to coordinates of the standard Aristotelian-Cartesian type $x^1 = x,\ x^2 = y,\ x^3 = z,\ x^0 = t,$ for which the metric takes the familiar form

$$d\tilde{s}^2 = dx^2 + dy^2 + dz^2 - \tilde{c}^2 dt^2 \,. \tag{35}$$

This particular time component will evidently be expressible with respect to an arbitrary coordinate system by the covariant formula

$$u_{\rm X}^{\,0} = u_{\rm X}^{\,\nu}\, t_\nu = c\big(\tilde{c}^2 - v_{\rm X}^{\,2}\big)^{-1/2}\,, \qquad v_{\rm X}^{\,2} = v_{\rm X}^{\,\nu}\, v_{{\rm X}\nu} = \eta_{\mu\nu} v_{\rm X}^{\,\mu} v_{\rm X}^{\,\nu}\,. \tag{36}$$

For the current itself we obtain the corresponding expression

$$n_{\rm X}^{\,\nu} = n_{\rm X}^{\,0}\left(e^\nu + v_{\rm X}^{\,\nu}\right)\,, \qquad n_{\rm X}^{\,0} = n_{\rm X} u_{\rm X}^{\,0} = n_{\rm X}^{\,\nu}\, t_\nu\,, \tag{37}$$

in which the relation between the rest frame number density $n_{\rm X}$ and the corresponding ether frame component $n_{\rm X}^{\,0}$ will be given by

$$n_{\rm X} = n_{\rm X}^{\,0}\, c^{-1}\big(\tilde{c}^2 - v_{\rm X}^{\,2}\big)^{1/2}\,. \tag{38}$$

so that we shall have

$$n_{\rm X}^{\,2} - n_{\rm X}^{\,0\,2} = n_{\rm X}^{\,0\,2}\left(\frac{2\phi}{c^2} - \frac{v_{\rm X}^{\,2}}{c^2}\right)\,. \tag{39}$$

We thereby obtain a decomposition of the form

$$\Lambda_{\rm bal} = \Lambda_{\rm ext} + \Lambda_{\rm rem}\,, \tag{40}$$

in which the extrinsic action contribution is given by

$$\Lambda_{\rm ext} = m\, \Sigma_{\rm X} \frac{n_{\rm X}^{\,0\,2}\big(v_{\rm X}^{\,2} - 2\phi\big)}{\big(n_{\rm X} + n_{\rm X}^{\,0}\big)}\,, \tag{41}$$

which is evidently well behaved in the Newtonian limit, while the remaining contribution will be expressible in terms of the total current (28) in the simple form

$$\Lambda_{\rm rem} = -m\, c^2 n^0\,, \qquad n^0 = n^\mu t_\mu\,. \tag{42}$$

Although this remainder $\Lambda_{\rm rem}$ will evidently be divergent in the large c limit, it can be seen that this does not matter from the point of view of the variational principle since the corresponding integrated action contribution, as given in terms of the standard coordinates used for (35) by

$$\mathcal{I}_{\rm rem} = \int \Lambda_{\rm rem}\, c^{-1} \tilde{\varepsilon}_{1\,2\,3\,0}\, dx\, dy\, dz\, dt, \tag{43}$$

will be expressible in terms of the global mass function (29) simply as

$$\mathcal{I}_{\rm rem} = -\int M\, c^2\, dt. \tag{44}$$

In the kind of application under consideration, the admissible variations must respect the relevant current conservation law (28), so that they will have no effect on the global mass function M, which will therefore just be a constant that may be taken outside the integration. This means that the remainder term (44) will have a fixed value, so that from the point of view of the variation principle its inclusion is entirely redundant. The ensuing theory is thus unaffected by replacing the original relativistic Lagrangian density $\tilde{\Lambda}$ by a new but equivalent Lagrangian density

$$\Lambda_{\rm new} = \tilde{\Lambda} - \Lambda_{\rm rem} = \Lambda_{\rm ext} + \Lambda_{\rm int}\,, \tag{45}$$

from which the divergent term has simply been subtracted out. The new version $\Lambda_{\rm new}$ has the technical disadvantage of being gauge dependent, since its specification depends on the choice of the ether frame characterised by the vector e^μ, but it has the important advantage of remaining well behaved in the Newtonian limit. With the usual assumption that (with respect to the chosen ether frame) the relevant squared space velocities and the potential are of the same order of smallness compared with the speed of light, we shall obtain

$$n_{\rm X} - n^{\,0}_{\rm X} = n_{\rm X}\left(\frac{\phi}{c^2} - \frac{v_{\rm X}^2}{2c^2}\right) + \mathcal{O}\left\{\frac{v^4}{c^4}\right\}, \tag{46}$$

as $c \to \infty$. It can thereby be seen that in this limit the new version of the action density will take the form

$$\Lambda_{\rm new} = \Lambda + \mathcal{O}\left\{\frac{v^4}{c^2}\right\}, \tag{47}$$

in which Λ is of purely Newtonian type, taking the standard generic form that was postulated at the outset in the preceding work [7, 8], namely

$$\Lambda = \Lambda_{\rm int} + \Lambda_{\rm ext}\,, \qquad \Lambda_{\rm ext} = \Lambda_{\rm kin} + \Lambda_{\rm pot}\,, \tag{48}$$

in which the kinetic and gravitational potential contributions have their usual Newtonian form

$$\Lambda_{\rm kin} = \tfrac{1}{2} m\, \Sigma_{\rm X} n_{\rm X} v_{\rm X}^2\,, \qquad \Lambda_{\rm pot} = -\phi\, m\, \Sigma_{\rm X} n_{\rm X}\,, \tag{49}$$

4 Noetherian construction of stress – energy tensor

For a variationally formulated theory in a general relativistic context, the corresponding stress momentum energy tensor is commonly constructed directly by differentiation of the relevant action density with respect to the spacetime metric. The purpose of the present section is to describe the adaptation of such a procedure to the technically more complicated Newtonian case.

In the preceding work on fluid systems [7, 8, 16] it was convenient to formulate the action in terms of physical fields (such as current 4-vectors) that are not entirely free but partially restrained as far as the application of the variation principle is concerned. However in the case of the solid systems with which the present article will be concerned it will be more convenient (albeit at the expense of greater gauge dependence) to work just with fields whose variation is unrestrained. The advantage of using an unrestrained variational formulation as we shall do here is that for an unperturbed field configuration satisfying the dynamical equations provided by the variational principle, the most general variation of the action will be unaffected, modulo the addition of a variationally irrelevant divergence, by the variations of the relevant dynamical fields, and will therefore be given, modulo such a divergence, just by the contributions from variations of the given background fields characterising the relevant Newtonian or relativistic spacetime structure.

In the relativistic case, the only independently given background field needed for this purpose is the metric $\tilde{g}_{\mu\nu}$ itself. Provided that the other dynamical fields in the Lagrangian $\tilde{\Lambda}$ obey the corresponding variational field equations, the generic action variation will be given by an expression of the simple form

$$\delta\tilde{\Lambda} \cong \frac{\partial\tilde{\Lambda}}{\partial\tilde{g}_{\mu\nu}}\,\delta\tilde{g}_{\mu\nu}\,. \tag{50}$$

using the symbol $\cong$ to denote equivalence modulo a divergence, i.e. modulo a term that is variationally irrelevant because its integral for a perturbation in a confined spacetime domain will vanish by Green's theorem.

As has long been well known in the context of general relativistic theory [14], and as has more recently been demonstrated in the Newtonian case [8], the use of a fully covariant formulation makes it possible to derive useful Noether type identities by considering variations of the trivial kind generated by an arbitrary displacement field, ξ^μ say. This means that the variation of each (background or dynamical) field variable will be given by the negative of its Lie derivative. In the relativistic case, the relevant variations will be given simply by

$$-\delta\tilde{\Lambda} = \vec{\xi}\pounds\,\tilde{\Lambda} \equiv \xi^\nu\tilde{\nabla}_\nu\tilde{\Lambda} \cong -\tilde{\Lambda}\tilde{\nabla}_\nu\xi^\nu\,, \tag{51}$$

and

$$-\delta\tilde{g}_{\mu\nu} = \vec{\xi}\pounds\,\tilde{g}_{\mu\nu} = 2\tilde{\nabla}_{(\mu}\xi_{\nu)}\,. \tag{52}$$

Their substitution in (50) provides a relation of the form

$$\tilde{T}^{\mu\nu}\tilde{\nabla}_{(\mu}\xi_{\nu)} \cong 0\,, \tag{53}$$

in which the relevant stress momentum energy density tensor can be read out as

$$\tilde{T}^{\mu\nu} = 2\frac{\partial\tilde{\Lambda}}{\partial\tilde{g}_{\mu\nu}} + \tilde{\Lambda}\,\tilde{g}^{\mu\nu}\,. \tag{54}$$

By again removing a variationally irrelevant divergence, (53) can be rewritten equivalently as

$$\xi^\mu\tilde{\nabla}_\nu\tilde{T}^\nu{}_\mu \cong 0\,, \tag{55}$$

which, since it must hold for a locally arbitrary vector field ξ^ν, shows that the variational field equations entail – as a generic consequence – a divergence condition of the well known form

$$\tilde{\nabla}_\nu\tilde{T}^\nu{}_\mu = 0\,, \tag{56}$$

which in the absence of gravity, i.e. in a flat Minkowski background, is interpretable as an energy-momentum conservation law.

The (automatically symmetric) "geometric" kind of stress-energy tensor (54) needs to be distinguished from the (related, but in general different and not necessarily symmetric) kind of stress-energy tensor commonly referred to as "canonical", which is constructed by differentiation not with respect to the metric, but with respect to the other dynamically relevant fields. Even in a special relativistic context, i.e. when gravitation is negligible so that the physical metric can be taken to be fixed, it is still perfectly legitimate (though the possibility of doing so is not widely realised) to exploit the greater convenience of the "geometric" construction via the consideration of virtual ("off shell") variations of the metric. However the construction of a "geometric" stress energy tensor is not quite so straightforward in a non-relativistic Newtonian framework, due to the degeneracy of the metric, which makes it harder to avoid the inelegancies of the traditional "canonical" treatment.

Although it is not quite so simple and convenient as in the relativistic case, a "geometric" type ansatz for the construction of an appropriate variational stress-energy tensor can nevertheless be obtained in a Newtonian framework using the 4-dimensionally covariant formalism set up in the previous section. Such an ansatz was developed in the preceding work [8], where it was shown how, in the case of a simple or multiconstituent fluid model, the action density function Λ for a 4-dimensionally covariant variational formulation provides a Noether identity that leads automatically to a natural "geometric" type ansatz for a corresponding non-relativistic stress-momentum-energy density tensor $T^{\mu}{}_{\nu}$. The present section describes the way to obtain the appropriate Noetherian ansatz for the non-relativistic "geometric" stress-energy tensor in a manner that is rather simpler than was required for the partially restrained case dealt with in the preceding work [8].

The kind of system under consideration is one governed by a non relativistic action integral of the form

$$\mathcal{I} = \int \Lambda \, \underline{\varepsilon}_{1230} \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3 \, \mathrm{d}x^0 \,, \tag{57}$$

in which the action density Λ is a scalar of the generic form (65) that is formulated as a function just of the relevant (variationally unrestrained) dynamical fields and their gradients, and of the background fields t_μ and $\gamma^{\mu\nu}$ characterising the Milne structure of Newtonian spacetime, as well as on a gauge dependent ether frame vector field e^μ. The involvement of the latter will violate the Milne and even Galileian invariance of the local action density, but will not violate the required covariance of the ensuing field equations provided the effect of an ether gauge transformation $\Lambda \mapsto \check{\Lambda}$ is just to add on a pure divergence term, which will not contribute to the integral of a local variation. This requirement is conveniently expressible as $\check{\Lambda} \cong \Lambda$, again using the symbol $\cong$ to denote equivalence modulo a (variationally irrelevant) divergence, for whose construction (due to the ether frame invariance of the measure tensor $\underline{\varepsilon}_{\mu\nu\rho\sigma}$ as remarked at the end of the preceding section) it makes no difference whether we use the original covariant differentiation operator, ∇, or the modified operator $\check{\nabla}$.

As remarked above, the simplification in the unrestrained case dealt with here is that for an unperturbed field configuration satisfying the dynamical equations provided by the variational principle, the most general variation of the action will be unaffected, modulo the addition of a variationally irrelevant divergence, by the variations of the relevant dynamical fields. Modulo such a divergence, the local action variation will therefore be given, in the non-relativistic case, just by the contributions from variations of the uniform background fields $\gamma^{\mu\nu}$, t_μ and e^μ, as well as of a generically non-uniform gravitational potential ϕ in cases for which the latter is introduced as given non dynamical background. The variation will therefore be given, modulo a divergence, in terms just of a set of tensorial coefficients $\partial\Lambda/\partial\gamma^{\mu\nu}$, $\partial\Lambda/\partial t_\mu$, and $\partial\Lambda/\partial e^\mu$ (of which the latter would not be needed in a model with ether gauge independent action density) by an expression of the form

$$\delta\Lambda \cong \frac{\partial\Lambda}{\partial\gamma^{\mu\nu}}\,\delta\gamma^{\mu\nu} + \frac{\partial\Lambda}{\partial t_\mu}\,\delta t_\mu + \frac{\partial\Lambda}{\partial e^\mu}\,\delta e^\mu + \frac{\partial\Lambda}{\partial\phi}\delta\phi\,, \tag{58}$$

It is to be remarked that this expression is not by itself sufficient to fully determine the coefficients involved. Further suitably chosen conventions, of which the most obviously appropriate is that the tensor $\partial\Lambda/\partial\gamma^{\mu\nu}$ should be symmetric, in view of the algebraic restrictions on the independence of the variations involved, which by (1) and (5) must must evidently satisfy

$$t_\mu\delta\gamma^{\mu\nu} = -\gamma^{\mu\nu}\delta t_\mu\,, \quad t_\mu\delta e^\mu = -e^\mu\delta t_\mu\,. \tag{59}$$

For the actual evaluation of the coefficients in (58) it will also be useful to have formulae for the variations of other associated spacetime background fields that may be involved in the explicit formulation of the action, such as the covariant spacetime metric whose variation will be given by the formula

$$\delta\eta_{\mu\nu} = -\eta_{\mu\rho}\eta_{\nu\sigma}\delta\gamma^{\rho\sigma} - 2t_{(\mu}\eta_{\nu)\lambda}\delta e^\lambda\,, \tag{60}$$

and the spacetime alternating tensor whose variation will be given by the formula

$$\delta\overline{\varepsilon}^{\mu\nu\rho\sigma} = \overline{\varepsilon}^{\mu\nu\rho\sigma}\left(\tfrac{1}{2}\eta_{\kappa\lambda}\delta\gamma^{\kappa\lambda} - e^\lambda\delta t_\lambda\right)\,, \tag{61}$$

(in which the ether frame dependence of the two separate terms can be seen to cancel out to give a gauge invariant total).

As in the relativistic case above, we now consider the effect of variations of the trivial kind generated by an arbitrary displacement field, ξ^μ say, so that the variation of each (background or dynamical) field variable will be given by the negative of its Lie derivative. The relevant formulae are thus given by

$$-\delta\Lambda = \vec{\xi}\pounds\,\Lambda \equiv \xi^\rho\nabla_\rho\Lambda\,, \tag{62}$$

$$-\delta\gamma^{\mu\nu} = \vec{\xi}\pounds\,\gamma^{\mu\nu} \equiv \xi^\rho\nabla_\rho\gamma^{\mu\nu} - 2\gamma^{\rho\,(\mu}\nabla_\rho\xi^{\nu)}\,, \tag{63}$$

$$-\delta t_\mu = \vec{\xi}\pounds\,t_\mu \equiv \xi^\rho\nabla_\rho t_\mu + t_\rho\nabla_\mu\xi^\rho\,, \tag{64}$$

$$-\delta e^\mu = \vec{\xi}\pounds\,e^\mu \equiv \xi^\rho\nabla_\rho e^\mu - e^\rho\nabla_\rho\xi^\mu\,. \tag{65}$$

$$-\delta\phi = \vec{\xi}\pounds\,\phi \equiv \xi^\rho\nabla_\rho\phi\,, \tag{66}$$

The first of these formulae can be rewritten, modulo a divergence, as

$$\delta\Lambda \cong \Lambda\nabla_{\!\rho}\xi^{\rho}\,, \tag{67}$$

while in view of the uniformity properties (4) and (6) of the unperturbed background fields, the next three will reduce to the form

$$\delta\gamma^{\mu\nu} = 2\gamma^{\rho\,(\mu}\nabla_{\!\rho}\xi^{\nu)}\,, \tag{68}$$

$$\delta t_{\mu} = -t_{\rho}\nabla_{\!\mu}\xi^{\rho}\,, \tag{69}$$

$$\delta e^{\mu} = e^{\rho}\nabla_{\!\rho}\xi^{\mu}\,. \tag{70}$$

For such a displacement variation, the relation (58) will therefore reduce to the form

$$T^{\mu}_{\ \nu}\nabla_{\!\mu}\xi^{\nu} \cong \rho\,\xi^{\nu}\nabla_{\!\nu}\phi\,, \tag{71}$$

in terms of a stress-momentum energy density tensor $T^{\mu}_{\ \nu}$ a gravitational mass density ρ that can be read out as

$$T^{\mu}_{\ \nu} = \Lambda\delta^{\mu}_{\nu} - 2\frac{\partial\Lambda}{\partial\gamma^{\rho\nu}}\,\gamma^{\rho\mu} + \frac{\partial\Lambda}{\partial t_{\mu}}\,t_{\nu} - \frac{\partial\Lambda}{\partial e^{\nu}}\,e^{\mu}\,, \tag{72}$$

and

$$\rho = -\frac{\partial\Lambda}{\partial\phi}\,. \tag{73}$$

By a further divergence adjustment the equivalence (71) can be rewritten as

$$\xi^{\nu}\big(\nabla_{\!\mu}T^{\mu}_{\ \nu} + \rho\nabla_{\!\nu}\phi\big) \cong 0\,, \tag{74}$$

which means that for a displacement confined to a localised spacetime region outside which the hypersurface contribution provided via Green's theorem by the unspecified divergence term will vanish) we shall have

$$\int \xi^{\nu}\big(\nabla_{\!\mu}T^{\mu}_{\ \nu} + \rho\nabla_{\!\nu}\phi\big)\,\varepsilon_{0123}\,\mathrm{d}x^{0}\,\mathrm{d}x^{1}\,\mathrm{d}x^{2}\,\mathrm{d}x^{3} = 0\,. \tag{75}$$

Since this identity must hold for an arbitrarily chosen displacement field ξ^{μ} in the spacetime region under consideration, it follows that we must have

$$\nabla_{\!\mu}T^{\mu}_{\ \nu} = -\rho\nabla_{\!\nu}\phi\,, \tag{76}$$

as a Noether type identity. We have thus established a theorem to the effect that the conservation law (76) will hold automatically for the geometric energy momentum tensor obtained from the covariantly formulated Newtonian action density Λ according to the prescription (72) whenever the dynamical field equations provided by the corresponding unrestrained variation principle are satisfied.

In the simple applications to be considered below in the present article, the local energy momentum conservation law (76) will, by itself amount to complete set of dynamical field equations, but it will only be a subset theroff in more general cases, such as the multiconstituent applications that we plan to deal with in subsequent work.

5 The material projection concept

The historical development of the standard textbook theory (see e.g. Landau and Lifshitz [18]) of a perfectly elastic solid in a Newtonian context is attributable to many people, among whom Cauchy is perhaps the most notable. However, as in the multiconstituent fluid case [5, 7], the insight needed for the formulation of a generally covariant version of the theory comes rather from its relativistic generalisation, for which a fully covariant formulation has always been used, as an indispensible necessity, from the outset. Some of the earliest work on the appropriate relativistic theory of a simple perfect solid was carried out in a purely mathematical context by Souriau [19], and by DeWitt [20] (who needed it as a toy model for testing techniques to be used in the quantisation of gravity). Its development as a realistic physical theory for use in the kind of astrophysical context (particular that of neutron stars) that motivates the present work was initiated (in the aftermath of the discovery of pulsars) with Quintana by one of the present authors [21], in a formulation [22] that was shown to be elegantly obtainable by a variational approach [15, 16] that will be used as a guide for the present work, whose purpose is to derive its Newtonian analogue.

The material projection is based on the simple consideration that the intrinsic structure of a solid is essentilly 3 dimensional. This means that the mathematical entities (such as differential forms) that will be used in a variational principle governing the dynamic behaviour of the solid should be defined over a 3 dimensional manifold, $\mathcal{X}$ say. The prototypical example of such field is the elastic-stress tensor, whose definition should not depend on the solid's history, whereas its explicit value obviously will do so. The requisite 3 manifold $\mathcal{X}$ is identifiable as the quotient of spacetime by the worldines of the idealised particles (representing microscopic lattice sites) constituing the solid , so that each point on $\mathcal{X}$ can be considered as the projection of the world-line describing the extrinsic motion of the relevant particle. Thus a patch of local coordinates, let us say q^A (for A =1,2,3), on $\mathcal{X}$ will induce a corresponding set of scalar fields q^A that will be given as functions of the local spacetime coordinates, x^μ (for $\mu = 0, 1, 2, 3$), on $\mathcal{M}$.

The local scalar fields q^A can be interpreted as a set of comoving – Lagrange type – coordinates on spacetime. They might even be used to specify the choice of the space coordinates by taking $x^1 = q^1$, $x^2 = q^2$ $x^3 = q^3$. However such a choice is more likely to be convenient in a general relativistic context than in that of a flat Minkowski background, or in the Newtonian case, for which a more commonly convenient choice is to use background coordinates x^μ flat (respectively Minkowski or Aristotelian-Cartesian) in order to simplify the procedure of covariant differentiation (in a manner that is not possible in the general relativistic case) by setting the connection coefficients $\Gamma_{\mu\ \rho}^{\ \nu}$ everywhere to zero.

Since the whole worldline of the particle is mapped into a point in $\mathcal{X}$ the q^A, when viewed as scalar fields on $\mathcal{M}$, will obviously be characterised by the property

$$\vec{u}\pounds q^A = u^\mu \nabla_\mu q^A = u^\mu q^A_{\ ,\mu} = 0\,, \tag{77}$$

where the tangent vector field u^μ to the worldline is subject to the standard normalisation condition given by $u^\mu u_\mu = -c^2$ in the relativistic case, and hence by $u^\mu t_\mu = 1$ in the Newtonian limit (24). Using the symmetry property (2) of the connection, which ensures that we shall have $\nabla_\mu q^A_{\ ,\nu} = \nabla_\nu q^A_{\ ,\mu}$, the relation (77) in turns implies that

$$\vec{u}\pounds q^{A}_{,\mu} = u^{\nu}\nabla_{\nu} q^{A}_{,\mu} + q^{A}_{,\nu}\nabla_{\mu} u^{\nu} = 0 \,. \tag{78}$$

Let us now consider the example of a material 1-form on the manifold $\mathcal{X}$, with components A_B say. When pulled back via the material projection, this material form induces a spacetime covariant 1-form on the spacetime manifold $\mathcal{M}$ with components A_μ given simply by

$$A_\mu = A_B q^{B}_{,\mu} \,, \tag{79}$$

with the direct implication that one will have

$$A_\mu u^\mu = 0 \,, \qquad \vec{u}\pounds A_\mu = 2u^\nu \nabla_{[\nu} A_{\mu]} = 0 \tag{80}$$

for any such material 1-form. Conversely if any spacetime 1-form is such that it satisfies both of the conditions above, then it is uniquely determined by a material 1-form through the pullback operation.

The generic defining property of the kind of simple perfectly elastic model to be considered here is that the action should depend only on the rheometric position coordinates q^A and on the corresponding induced contravariant metric components γ_{AB}, or equivalently on the corresponding contravariant components γ^{AB}, which are defineable by the reciprocity relation

$$\gamma^{AB}\gamma_{BC} = \delta^{A}_{C} \,, \tag{81}$$

and which will be given in the relativistic case simply as the rheometric projection of the contravariant spacetime metric,

$$\gamma^{AB} = \tilde{g}^{\mu\nu} q^{A}_{,\mu} q^{B}_{,\nu} \,, \tag{82}$$

while in the Newtonian limit (24) the required components will be given by

$$\gamma^{AB} = \gamma^{\mu\nu} q^{A}_{,\mu} q^{B}_{,\nu} \,. \tag{83}$$

It is to be remarked that, in solid models of the kind appropriate for typical (low pressure) laboratory type terrestrial applications, the rest-frame energy per particle will commonly have an absolute minimum for some preferred value, κ_{AB} say, of the induced metric γ_{AB}. In such such cases κ_{AB} can be considered, and exploited, as a natural fixed (positive definite) Riemannian metric on the material base space. In other words $\kappa_{AB}\,\mathrm{d}q^A\mathrm{d}q^B$ represents the "would be" relaxed distance between the chosen particle and a nearby one in the frame of the former, assuming the attainability of such relaxed state (which could be defined by extracting a piece of the continuous medium – i.e. a neighbourhood of the considered particle – and letting it reach the minimum local energy density state). In generic circumstances however, such a preferred rheometric metric may be ill defined, since a local state of minimised energy need not exist. This caveat applies particular in cases of very high pressure (such as will occur in deep stellar interiors) from which a process relaxation might lead, not to a minimised energy state for the solid, but merely to its vaporisation as a gas.

The formula (83), while defined in the Newtonian limit, will also be valid in the relativistic case provided it is interpreted in terms of the relevant rank-3 worldline-tangential projection tensor, which will be given by

$$\gamma^{\mu}_{\nu} = \delta^{\mu}_{\nu} + c^{-2} u^{\mu} u_{\nu} \,, \tag{84}$$

in the relativistic case, and which will will go over, in the Newtonian limit (24), to

$$\gamma^{\mu}_{\nu} = \delta^{\mu}_{\nu} - u^{\mu} t_{\nu} \,. \tag{85}$$

(Except in the case of a static configuration – for which the ether vector e^{μ} may be chosen to coincide with u^{μ} – the variable flow-tangential projection tensor defined by (85) must be distinguished from the uniform ether frame dependent projection tensor (5) that was denoted by the same symbol in the preceding work [7, 8] but that is denoted here by η^{μ}_{ν}).

We shall use the induced base metric of (81) for raising and lowering of the material base indices in the usual way, as illustrated for a covector with components $A_{_A}$ by

$$A^{A} = \gamma^{AB} A_{_B} \,, \quad A_{_A} = \gamma_{_{AB}} A^{B} \,, \tag{86}$$

The (variable) covariant metric as defined by

$$\gamma_{\mu\nu} = \gamma_{_{AB}} q^{A}_{\ ,\mu} q^{B}_{\ ,\nu} \,. \tag{87}$$

can be used in conjunction with the (uniform) contravariant metric $\gamma^{\mu\nu}$ for an unambiguously reversible index raising lowering and raising operation for vectors and tensors that are orthogonal to the material flow in the manner exemplified, according to (80), by the pull back (79) of $A_{_A}$ namely

$$A_{\mu} = A_{_A} q^{A}_{\ \mu} = \gamma_{\mu\nu} A^{\nu} \,, \tag{88}$$

whose raised version will project onto A^{A}:

$$A^{\mu} = \gamma^{\mu\nu} A_{\nu} \,, \quad A^{\mu} t_{\mu} = 0 \,, \quad q^{A}_{\ \mu} A^{\mu} = A^{A} \,. \tag{89}$$

Is to be noted that although the procedure defined by (86) is invariant with respect to the choice of ether frame, it does depend on time. Thus if $A_{_A}$ is taken to be a fixed vector potential characterising the "frozen-in" magnetic field that will be introduced below, so that its time derivative $\dot{A}_{_A}$ will vanish, the corresponding contravariant vector will nevertheless be time dependent:

$$\dot{A}_{_A} = 0 \ \Rightarrow \ \dot{A}^{A} = \dot{\gamma}^{AB} A_{_B} \,. \tag{90}$$

It can be seen from (4) and (78) that the time derivative, along the worldlines, of the induced base metric will be given by

$$\dot{\gamma}_{_{AB}} = 2\theta_{_{AB}} = -\gamma_{_{AC}} \gamma_{_{BD}} \dot{\gamma}^{CD} \,, \quad \dot{\gamma}^{AB} = -2\theta^{AB} = -2 q^{A}_{\ ,\mu} q^{B}_{\ ,\nu} \theta^{\mu\nu} \,, \tag{91}$$

in terms of the (symmetric) strain rate tensor

$$\theta^{\mu\nu} = \gamma^{\rho(\mu} \nabla_{\rho} u^{\nu)} \,. \tag{92}$$

which will automatically satisfy the orthogonality condition $\theta^{\mu\nu} u_{\nu} = 0$ in the relativistic case, so that we shall have $\theta^{\mu\nu} t_{\nu} = 0$ in the Newtonian limit.

The time derivation considered so far has concerned only quantities such as the base space components that have the status of scalars from the point of view of the background space time. We now extend the dot notation to quantities that are tensorial with respect to the background spacetime by defining it to indicate covariant differentiation with respect to time along the world lines, meaning that will indicate the effect of the operator $u^\rho\nabla_\rho$ in the manner illustrated by the definition of the acceleration vector which will be given by

$$\dot{u}^\mu = u^\rho\nabla_\rho u^\mu \,. \tag{93}$$

so that it will satisfy the condition $\dot{u}^\mu u_\mu = 0$ in the relativistic case and $\dot{u}^\mu t_\mu = 0$ in the Newtonian limit. It can be seen that the corresponding covariant time derivatives $\dot{\gamma}_{\mu\nu} \equiv u^\rho\nabla_\rho\gamma_{\mu\nu}$ of the gauge invariant metric fields $\gamma_{\mu\nu}$ defined by (92) will be given in the relativistic case by

$$\dot{\gamma}_{\mu\nu} = \frac{2}{c^2}u_{(\mu}\dot{u}_{\nu)} \,, \tag{94}$$

and in the Newtonian limit by

$$\dot{\gamma}_{\mu\nu} = -2t_{(\mu}\gamma_{\nu)\rho}\dot{u}^\rho \,, \quad \dot{\gamma}^\mu_\nu = -t_\nu\dot{u}^\mu \,. \tag{95}$$

It is useful for many purposes – and will be indispensible for the discussion of the Newtonian limit - to introduce an appropriate fixed measure form, with antisymmetric components $N_{_{ABC}} = N_{[_{ABC}]}$ say, on the rheometric base manifold. Such a measure will typically be interpretable as representing the density of microscopic lattice points in an underlying cristal structure. Such a measure will determine a corresponding contravariant spacetime current of the kind introduced in (27) by a pull back relation of the form

$$n^*_{\mu\nu\rho} = N_{_{ABC}}\, q^{_A}_{,\mu} q^{_B}_{,\nu}\, q^{_C}_{,\rho} \,, \tag{96}$$

The corresponding scalar number density n in the rest frame of the medium will evidently be given by

$$n^2 = \frac{1}{3!}N_{_{ABC}}N_{_{DEF}}\,\gamma^{_{AD}}\gamma^{_{BE}}\gamma^{_{CF}} \,. \tag{97}$$

It is then obvious that the current density so defined will be automatically conserved due to the closure of the 3 form $N_{_{ABC}}$ (being a 3 form on a 3 manifold):

$$\nabla_{[\lambda}N_{\mu\nu\rho]} = 0 \,, \quad \nabla_\mu n^\mu = 0 \,. \tag{98}$$

In the simple purely elastic application considered here, the formalism defined above will be enough to describe the dynamics of the solid. However such a simplification will not be possible in the more general applications envisaged for future work, such as transfusive exchange [2] of matter between distinct chemical constituents, in which one no longer has conservation of the solid's current.

6 Action for a simple perfectly elastic solid

Whereas the internal energy depends only on the density when a simple fluid is considered, in the case of a solid it will also be necessary to allow for the reaction of the internal energy not just to changes of volume, but also to changes in shear strain. This is done by allowing Λ_{int} to depend on γ^{AB}, which, because of its time dependence can be interpreted as a Cauchy type strain tensor. This kind of (perfectly elastic) solid model includes the category of simple (barotropic) perfect fluid models as the special case for which the dependence is only on the determinant of γ^{AB}. For a medium that is perfectly elastic in this sense, the generic action variation will be given in the relativistic case by the formula

$$\delta\tilde{\Lambda} = \frac{\partial\tilde{\Lambda}}{\partial\gamma^{AB}}\delta\gamma^{AB} + \frac{\partial\tilde{\Lambda}}{\partial q^{A}}\delta q^{A} , \tag{99}$$

which characterises the partial derivative components needed for the specification of the rheometric stress tensor S_{AB} and its spacetime pullback

$$S_{\mu\nu} = S_{AB}\, q^{A}_{\ ,\mu} q^{B}_{\ ,\nu} , \quad S_{AB} = 2\frac{\partial\tilde{\Lambda}}{\partial\gamma^{AB}} - \tilde{\Lambda}\gamma_{AB} . \tag{100}$$

It can be seen from (54) and (84) that the complete stress energy tensor will be given in terms of this quantity by an expression of the standard form

$$\tilde{T}^{\mu\nu} = \tilde{\rho}\, u^{\mu} u^{\nu} - S^{\mu\nu} , \tag{101}$$

in which comoving mass-density density $\tilde{\rho}$ is given by the simple proportionality relation

$$\tilde{\Lambda} = -\tilde{\rho}\, c^{2} . \tag{102}$$

This is all that is needed for the formulation of the variational field equations, which in this case are given completely just by the Noetherian divergence condition (56).

In typical physical applications the mass-energy function in (102) will have a minimum value $\rho_{\rm flue}$ – corresponding to a maximum value $\Lambda_{\rm flu}$ of the elastic action function $\tilde{\Lambda}$ – for any given value of the determinant $|\gamma|$ of the induced metric γ_{AB}, or equivalently for any given value of the number density n as specified by (97). This means that the action will be decomposible in the form

$$\tilde{\Lambda} = \Lambda_{\rm flu} + \Lambda_{\rm rig} , \tag{103}$$

in which the – usually rather small – remainder term – $\Lambda_{\rm rig}$ is a negative indefinite rigidity contribution contribution without which the medium would be of purely fluid type. The – usually dominant – perfect fluid contribution $\tilde{\Lambda}_{\rm flu}$ will itself be itself decomposible in the form

$$\Lambda_{\rm flu} = \Lambda_{\rm bal} + \Lambda_{\rm pre} , \tag{104}$$

in which $\Lambda_{\rm pre}$ is a pressure energy contribution that in typical applications will also be relatively small compared with a dominant ballistic contribution $\Lambda_{\rm bal}$ given by an appropriate choice of the mass parameter m in the general formula (32) which, in the single constituent case considered here, reduce to the trivial form

$$\Lambda_{\rm bal} = -\rho\, c^2 , \qquad \rho = m\, n, \tag{105}$$

so that the mass density (102) will be expressible in the form

$$\tilde\rho = \rho + c^{-2}\epsilon\,, \tag{106}$$

with the relatively small internal energy contribution given according to (31) by

$$\epsilon = -\Lambda_{\rm int}\,. \tag{107}$$

The action decomposition (103) corresponds to an energy density decomposition of the form

$$\epsilon = \epsilon_{\rm pre} + \epsilon_{\rm rig}\,, \tag{108}$$

in which the rigidity contribution $\epsilon_{\rm rig}$ will vanish when ϵ is minimised for a given value of n. The non trivial pressure term

$$\Lambda_{\rm pre} = -\epsilon_{\rm pre}\,, \tag{109}$$

in (104) will evidently be a function just of the undifferentiated base space coordinates q^A and of the scalar density n, which means that, as the analogue of (99), its generic variation will be given by

$$\delta\Lambda_{\rm pre} = -\frac{\partial\epsilon_{\rm pre}}{\partial n}\delta n - \frac{\partial\epsilon_{\rm pre}}{\partial q^A}\delta q^A\,, \tag{110}$$

with

$$\delta n = \frac{n}{2}\gamma_{AB}\,\delta\gamma^{AB}\,. \tag{111}$$

It is evident from (100) that the ballistic term will make no contribution at all to the strain tensor, while the contribution from the pressure energy term will of course be purely isotropic:

$$S_{\rm bal}^{\ \mu\nu} = 0\,, \quad S_{\rm pre}^{\ \mu\nu} = -P_{\rm pre}\gamma^{\mu\nu}\,, \quad P_{\rm pre} = n\frac{\partial\epsilon_{\rm pre}}{\partial n} - \epsilon_{\rm pre}\,. \tag{112}$$

This means that the purely intrinsic action contribution given, according to (108), by

$$\Lambda_{\rm int} = \Lambda_{\rm pre} + \Lambda_{\rm rig}\,, \tag{113}$$

will be sufficient by itself to determine the the complete stress tensor in (101), which will take the form

$$S^{\mu\nu} = S_{\rm int}^{\ \mu\nu} = -P_{\rm pre}\gamma^{\mu\nu} + S_{\rm rig}^{\ \mu\nu}\,. \tag{114}$$

It is important to notice that in this simple elastic case the effect of a worldline displacement along the worldlines will have no effect so substitution of u^μ for ξ^μ in (55) will merely give an identity

$$u^\mu\tilde\nabla_\nu\tilde T^{\ \nu}_{\mu} = 0\,. \tag{115}$$

A complete set of dynamical equations will therefore be provided just by the orthogonal projection of (56) which will be expressible as

$$\big(\tilde\rho\gamma^\mu_{\ \nu} - c^{-2}S^\mu_{\ \nu}\big)\dot u^\nu = \gamma^\mu_{\ \rho}\gamma^{\sigma\nu}\tilde\nabla_\nu S^\rho_{\ \sigma}\,. \tag{116}$$

7 Newtonian dynamics of a simple perfectly elastic solid

To obtain the Newtonian limit of the simple relativistic elasticity model characterised by the stress energy tensor (101) that was set up in the preceding section, it is now straightforward to apply the procedure described in Section 3, according to which we should use a Newtonian action integral

$$\mathcal{I} = \int \Lambda\, \underline{\varepsilon}_{1230}\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3\, \mathrm{d}x^0\,, \tag{117}$$

with a Lagrangian of the form (48), which in this simple elastic case reduces just to

$$\Lambda = \Lambda_{\text{kin}} + \Lambda_{\text{pot}} - \epsilon\,, \tag{118}$$

with an internal energy function ϵ of the same form as in the relativistic case, and with the Newtonian kinetic and potential energy terms given by expressions of the same form as for a simple fluid, namely

$$\Lambda_{\text{kin}} = \tfrac{1}{2}\rho\, v^2\,, \quad \Lambda_{\text{pot}} = -\phi\rho\,, \tag{119}$$

in which the 3-velocity v^μ is characterised in terms of the chosen ether frame frame, e^μ by

$$v^\mu = u^\mu - e^\mu\,, \quad v^2 = \eta_{\mu\nu}\, u^\mu u^\nu\,, \tag{120}$$

and relevant Newtonian mass density will be given simply by

$$\rho = m\, n\,, \tag{121}$$

for a number density n that, in accordance with (96), will be expressible in this limit as

$$n = n^\mu t_\mu = \frac{1}{3!}\epsilon^{\mu\nu\rho} n_{\mu\nu\rho}\,, \quad n^\mu = n u^\mu = \frac{1}{3!}\overline{\varepsilon}^{\mu\nu\rho\sigma} n^*_{\nu\rho\sigma}\,. \tag{122}$$

The current 3 form $n^*_{\nu\rho\sigma}$ itself is obtained from the fixed 3- form N_{ABC} on the material base space by the construction (96), which automatically ensures the satisfaction of the non-relativistic conservation law in its usual form

$$\nabla_\mu n^\mu = 0\,. \tag{123}$$

It is to be remarked that, as in the fluid case [7], the gravitational coupling term Λ_{pot} will be unaffected by linear (Galilean) gauge transformations but not by accelerated Milne transformations, while the kinetic term Λ_{kin} is of course even more highly frame dependent (not even Galilei invariant). For the purpose of comparison with the preceding work, the terms in the Newtonian action can usefully be regrouped in the standard form

$$\Lambda = \Lambda_{\text{mat}} + \Lambda_{\text{pot}} \quad \Lambda_{\text{mat}} = \Lambda_{\text{kin}} - \epsilon\,, \tag{124}$$

In the perfect fluid case this elastic energy density ϵ will be given on each worldline (as specified by the values q^A) as a corresponding function (in the barotropic case [7] everywhere the same function) of the number density n, which can be seen from (121) to be given as a function of the scalar product fields γ^{AB} by the determinant formula

$$n^2 = \frac{1}{3!} N_{ABC} N_{DEF} \gamma^{AD} \gamma^{BE} \gamma^{CF} \, . \tag{125}$$

The generalisation from a perfect fluid to a perfectly elastic solid is made simply by taking $\in$ to be a generic (worldline dependent) function of the scalar product fields γ^{AB} (i.e. of the tensorial projection of $\gamma^{\mu\nu}$ onto $\mathcal{X}$) instead of being restricted to the depend just on the determinental combination (125) as in the fluid case. This means allowing $\in$ to be affected not just by changes of volume but also by shearing strain, whose effect is allowed for by supplementing the purely fluid contribution $\in_{\rm pre}$ in (107) by the extra rigidity term $\in_{\rm rig}$.

Using (60) and (61) we obtain the formulae

$$\delta u^\mu = -u^\mu u^\nu \delta t_\nu \tag{126}$$

and

$$\delta n = \tfrac{1}{2} n \gamma_{\mu\nu} \delta\gamma^{\mu\nu} = n \eta_{\mu\nu} \left(\tfrac{1}{2} \delta\gamma^{\mu\nu} + u^\mu \gamma^{\nu\rho} \delta t_\rho \right) \tag{127}$$

for variations in which the fields q^A are held constant – meaning that the world lines remain fixed – it can be seen from the defining ansatz (72) that the kinetic and gravitational potential contributions to the stress momentum energy density tensor will be given by expressions that are already familiar from experience [7] with the simple fluid case, namely

$$\underset{\rm kin}{T}{}^{\mu}_{\ \nu} = n^\mu p_\nu \tag{128}$$

in which the frame dependent 4- momentum per particle is given, using the notation of (120) by

$$p_\mu = m \left(v_\mu - \tfrac{1}{2} v^2 t_\mu \right) \tag{129}$$

and

$$\underset{\rm pot}{T}{}^{\mu}_{\ \nu} = -\phi \rho^\mu t_\nu \, , \quad \rho^\mu = m n^\mu . \tag{130}$$

The corresponding stress tensor S^{AB} is definable in the traditional manner in terms of the effect on the energy per particle $\in/n$ of an infinitesimal strain variation $\delta\gamma_{AB}$ according to a prescription of the form

$$n \, \delta \left(\frac{\in}{n} \right) = \tfrac{1}{2} S^{AB} \delta\gamma_{AB} . \tag{131}$$

By comparing this with the formula (111) for δn, it can be seen that this contravariant stress tensor will be expressible independently of the number density as

$$S^{AB} = 2 \frac{\partial \in}{\partial \gamma_{AB}} + \in \gamma^{AB} , \tag{132}$$

which means that its covariant (index lowered) version will be given by

$$S_{AB} = -2 \frac{\partial \in}{\partial \gamma^{AB}} + \in \gamma_{AB} \, . \tag{133}$$

In the manner described above this base tensor will determine a corresponding space stress tensor

$$S_{\mu\nu} = S_{AB}\, q^A{}_{,\mu} q^A{}_{,\nu}\,, \qquad S_{\mu\nu} u^\nu = 0\,, \tag{134}$$

in terms of which the rate of change of the energy density along the world lines will be given by

$$\dot{\epsilon} = S_{\mu\nu}\theta^{\mu\nu} - \epsilon\theta\,, \tag{135}$$

where θ is the expansion rate, as given by

$$\theta = \gamma_{\mu\nu}\theta^{\mu\nu} = \nabla_\mu u^\mu = -\dot{n}/n\,. \tag{136}$$

Using the formula

$$\frac{\partial \gamma^{AB}}{\partial \gamma^{\mu\nu}} = q^A{}_{,(\mu} q^B{}_{,\nu)} \tag{137}$$

obtained from (83), it can be now seen from (131) that we shall have

$$\frac{\partial \epsilon}{\partial \gamma^{\mu\nu}} = \tfrac{1}{2}(\epsilon\gamma_{AB} - S_{AB}) q^A_{,\mu} q^B_{,\nu} = \tfrac{1}{2}(\epsilon\gamma_{\mu\nu} - S_{\mu\nu})\,. \tag{138}$$

The relation (107) thus gives

$$\frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\mu\nu}} = \tfrac{1}{2}(S_{\mu\nu} - \epsilon\gamma_{\mu\nu})\,, \tag{139}$$

so that by the general formula (72)(since no dependence on e^μ or t_μ is involved in this case) the internal contribution to the stress momentum energy tensor will be given by

$$\underset{\text{int}}{T}{}^\mu{}_\nu = -\epsilon\delta^\mu_\nu + \gamma^{\mu\rho}(\epsilon\gamma_{\rho\nu} - S_{\rho\nu})\,, \tag{140}$$

which simplifies, using (85) and the (orthogonal) index raising operation exemplified by (88) to

$$\underset{\text{int}}{T}{}^\mu{}_\nu = -\epsilon u^\mu t_\nu - S^\mu_\nu\,. \tag{141}$$

Thus, combining this gauge independent internal contribution with the ether frame dependent kinetic contribution (128), we end up with the complete material stress energy tensor

$$\underset{\text{mat}}{T}{}^\mu{}_\nu = \underset{\text{kin}}{T}{}^\mu{}_\nu + \underset{\text{int}}{T}{}^\mu{}_\nu\,, \tag{142}$$

in the form

$$\underset{\text{mat}}{T}{}^\mu{}_\nu = n^\mu m_\nu - S^\mu_\nu\,. \tag{143}$$

in which the relevant (gauge dependent) momentum per particle covector m_μ is given by

$$m_\nu = p_\nu - (\epsilon/n)t_\nu = m\left(v_\nu - \left(\tfrac{1}{2}v^2 + \epsilon/\rho\right)t_\nu\right)\,. \tag{144}$$

Since it is not orthogonal to the flow this momentum 1-form is not completely determined just by the corresponding – purely kinematic – contravariant momentum covector, namely

$$m^\mu = \gamma^{\mu\nu} m_\nu = m v^\mu\,. \tag{145}$$

In the special case of the perfect fluid limit we shall simply have $S^{\mu\nu} = -P\gamma^{\mu\nu}$, where P is the ordinary scalar pressure. It is important to note, in this case, that the momentum covector m_ν introduced here will not be quite the same as the material momentum covector μ_ν that was introduced in the preceding work [7], since m_ν is defined in terms of the integrated internal energy per particle, namely $\in/n$, whereas μ_ν was defined in terms of the differential internal energy per particle, namely the chemical potential, $\chi = (\in + P)/n$. This means that, in the (barotropic) perfect fluid limit, the relevant material momentum 1-form will be given by $\mu_\mu = m_\mu - (P/n)t_\mu$.

The complete stress energy tensor, including allowance for the gravitational background, can now be obtained as

$$T^{\mu}_{\ \nu} = T_{\text{mat}}{}^{\mu}_{\ \nu} + T_{\text{pot}}{}^{\mu}_{\ \nu}\,, \tag{146}$$

which by (130) finally gives

$$T^{\mu}_{\ \nu} = n^{\mu}(m_\nu - m\phi t_\nu) - S^{\mu}_{\ \nu}\,. \tag{147}$$

for the tensor whose divergence will provide the required dynamical equations according to the Noether relation (76).

For a compound system (as exemplified by the multiconstituent fluid models studied in the preceding work [8]) the 4-independent components of the Noether relation (76) would not by themselves be sufficient to fully determine the dynamical evolution. However for a simple medium such as we are considering here, for which the dynamics are completely describable just in terms of the motion of the world lines, as given by the evolution of the 3 independent scalar fields q^A, the evolution is actually overdetermined by the 4 components of (76), whose contraction with 4-velocity, namely

$$u^\nu f_\nu = 0\,, \qquad f_\nu = (\nabla_\mu T^{\mu}_{\ \nu} + \rho\nabla_\nu\phi) = 0\,, \tag{148}$$

merely gives the kinematic identity (134), which must be satisfied as a mathematical necessity, even under circumstances more general than those considered here, in which the space components of the force density f_μ acting on the medium might not be zero. The underlying reason for this identity is that whereas the dynamical equations were needed for the derivation of the Noetherian condition (74) for an arbitrary displacement vector field ξ^ν, it would evidently hold as a mere identity for a displacement along the flow lines, i.e. for $\xi^\nu \propto u^\nu$, in the simple elastic case. It is to be remarked that the logic could be reversed as, was done in the cited work on the relativistic case, which started [21] by postulating the analogue of (148) as a condition needed for consistency, and then went on [15] to derive the action formulation as a consequence.

The upshot is that the complete system of dynamical evolution equations for a simple elastic solid model will be given just by the 3 independent components of the space projection of (76) namely

$$\gamma^{\mu\nu}\big(\nabla_\rho T^{\rho}_{\ \nu} + \rho\nabla_\nu\phi\big) = 0\,. \tag{149}$$

Using the conservation law (123), it can be seen from the formula (147) that in terms of the frame dependent gravitational field vector,

$$g^{\mu} = -\gamma^{\mu\nu}\nabla_\nu\phi \tag{150}$$

this system will be expressible in the form

$$\rho(\dot{u}^\mu - g^\mu) = \nabla_\nu S^{\mu\nu}\,, \tag{151}$$

in which it is to be observed that each side (though not the separate terms on the left) is satisfactorily invariant, not just under linear Galilean tranformations, but even under arbitrarily accelerated Milne transformations.

8 Derivation of the characteristic equation

As in the relativistic case [25], let us now seek the conditions governing the a covector λ_μ say that is normal to a characteristic hypersurface across which the relevant field quantities n, u^μ, and $S^{\mu\nu}$ have discontinuous gradients, using the standard method of Hadamard, which exploits the condition that the discontinuity of the gradient of a continuous scalar field must be proportional to the normal covector λ_μ, so that in particular for the density we shall have

$$[\nabla_\mu n] = \hat{n}\lambda_\mu\,, \tag{152}$$

for some corresponding scalar discontinuity amplitude $\hat{n}$. The associated unit propagation vector ν^μ characterised by

$$\gamma^{\mu\nu}\nu_\nu\nu_\mu = 1\,, \quad \nu_\mu u^\mu = 0\,, \tag{153}$$

and the propagation velocity, υ say, relative to the local rest frame, of the discontinuity are specifiable by taking λ_μ to have the standard normalisation so that it takes the form

$$\lambda_\mu = \nu_\mu + \upsilon\, c^{-2} u_\mu\,, \tag{154}$$

in the relativistic case, and hence

$$\lambda_\mu = \nu_\mu - \upsilon t_\mu\,, \tag{155}$$

in the Newtonian limit. In terms of the same discontinuity covector as in (152) the discontinuity of the gradient of u^μ will be given by an expression of analogous form,

$$[\nabla_\mu u^\nu] = \hat{u}^\nu\lambda_\mu\,, \tag{156}$$

in terms of a corresponding vectorial discontinuity amplitude $\hat{u}^\nu$, while similarly for the stress tensor we shall have

$$[\nabla_\mu S^{\nu\rho}] = \hat{S}^{\rho\nu}\lambda_\mu\,. \tag{157}$$

Since the evolution of n and $S^{\mu\nu}$ is kinematically determined by that of the flow lines, the corresponding gradient discontinuity amplitudes $\hat{n}$ and $\hat{S}^{\nu\rho}$ will be determined by the velocity gradient discontinuity amplitude $\hat{u}$. In the case of the number density n it can be seen from (136) that we shall have

$$u^\mu\nabla_\mu n = \dot{n} = -\rho\theta = -n\nabla_\mu u^\mu \tag{158}$$

so by taking the discontinuity we obtain

$$u^\mu\lambda_\mu\hat{n} = -n\lambda_\mu\hat{u}^\mu\,. \tag{159}$$

The normalisation conditions $u^\mu u_\mu = -c^2$ in the relativistic case and $u^\mu t_\mu = 1$ in the Newtonian case imply corresponding restrictions $\hat{u}^\mu u_\mu = 0$, $\hat{u}^\mu t_\mu = 0$ respectively, with the implication that for $\hat{u}^\mu$, as for ν_μ we can unambigously and reversibly raise and lower the indices by contraction with $\gamma^{\mu\nu}$ and $\gamma_{\mu\nu}$. It follows that (159) will reduce to the simple form

$$\upsilon\, \hat{n} = n\, \nu_\mu \hat{u}^\mu \, . \tag{160}$$

To write the corresponding relation for $S^{\mu\nu}$ we need the relevant elasticity tensor, which is defined in such a way as to have the symmetry properties

$$E^{ABCD} = E^{CDAB} = E^{(AB)(CD)}, \tag{161}$$

by the ansatz

$$E^{ABCD} = 4n \frac{\partial^2}{\partial\gamma_{AB}\partial\gamma_{CD}}\left(\frac{\epsilon}{n}\right) = 2\frac{\partial S^{AB}}{\partial\gamma_{CD}} + S^{AB}\gamma^{CD}. \tag{162}$$

In terms of this highly symmetric elasticity tensor, the less highly symmetric Hadamard elasticity tensor that will be needed below is specifiable as

$$A^{ABCD} = A^{CDAB} = E^{ABCD} + \gamma^{AC} S^{BD}. \tag{163}$$

It follows from (162) that the time derivative of the stress tensor in the material base space will be given in terms of that of the strain tensor by

$$\dot{S}^{AB} = \tfrac{1}{2}(E^{ABCD} - S^{AB}\gamma^{CD})\dot{\gamma}_{CD} \, . \tag{164}$$

For the purpose of evaluating the time derivatives of the corresponding space time tensors, contravariant base tensors are less convenient than the corresponding covariant tensors, whose time derivative can be seen, by (78), to pullback directly onto the corresponding Lie derivative in the manner illustrated in the case of the stress as

$$q^A_{\,,\mu} q^B_{\,,\nu} \dot{S}_{AB} = \vec{u}\pounds S_{\mu\nu} = u^\rho \nabla_\rho S_{\mu\nu} + 2S_{\rho(\mu}\nabla_{\nu)} u^\rho \, . \tag{165}$$

We therefore need the formula obtained by swapping covariant with contravariant indices in (164) which gives

$$\dot{S}_{AB} = -\tfrac{1}{2}(E_{ABCD} - S_{AB}\gamma_{CD} + 4S_{C(A}\gamma_{B)D})\dot{\gamma}^{CD} \, , \tag{166}$$

from which, by (91) we obtain

$$q^A_{\,,\mu} q^B_{\,,\nu} \dot{S}_{AB} = (E_{\mu\nu\rho\sigma} - S_{\mu\nu}\gamma_{\rho\sigma} + 4S_{\rho(\mu}\gamma_{\nu)\sigma})\theta^{\rho\sigma} \, . \tag{167}$$

Combining this with (165) and using the definition (92) of the expansion rate tensor $\theta^{\mu\nu}$ we obtain an evolution equation for the stress tensor in the form

$$u^\rho \nabla_\rho S_{\mu\nu} = -2S_{\rho(\mu}\nabla_{\nu)} u^\rho + \left(E_{\mu\nu\rho}{}^{\sigma} - S_{\mu\nu}\gamma_\rho{}^{\sigma} + 2S_{\rho(\mu}\gamma_{\nu)}{}^{\sigma} + 2S^\sigma{}_{(\mu}\gamma_{\nu)\rho}\right)\nabla_\sigma u^\rho \, . \tag{168}$$

Taking the discontinuity of the gradients in this relation we obtain, as the analogue of (159),

$$u^\rho \lambda_\rho \hat{S}_{\mu\nu} = -2S_{\rho(\mu}\lambda_{\nu)}\hat{u}^\rho + \big(E_{\mu\nu\rho}{}^\sigma - S_{\mu\nu}\gamma_\rho{}^\sigma + 2S_{\rho(\mu}\gamma_{\nu)}^{\ \sigma} + 2S^\sigma{}_{(\mu}\gamma_{\nu)\rho}\big)\lambda_\sigma \hat{u}^\rho \,. \tag{169}$$

After projecting out the time component by contraction with $\gamma^{\lambda\mu}$, this leaves, as the analogue of (160), an expression giving the stress gradient discontinuity amplitude as a function of the velocity gradient discontinuity in the form

$$\upsilon\,\gamma^{\mu\rho}\gamma^{\nu\sigma}\hat{S}_{\rho\sigma} = -\big(E^{\mu\nu\rho}{}_\sigma - S^{\mu\nu}\gamma^\rho_\sigma + 2S_\sigma^{\ (\mu}\gamma^{\nu)\rho}\big)\nu^\sigma \hat{u}_\rho ,\,. \tag{170}$$

We now have all that is needed for processing the gradient discontinuity relation provided by the equations of motion, from which one obtains the dynamical equation

$$\rho u^\rho \lambda_\rho\, y^\mu_\nu \hat{u}^\nu = \lambda_\nu \gamma^{\mu\rho}\gamma^{\nu\sigma}\hat{S}_{\rho\sigma}\,, \tag{171}$$

in terms of a tensor y^μ_ν that will be given in the relativistic case (116) by the relation

$$\rho c^2 y^\mu_\nu = (\rho c^2 + \epsilon)\gamma^\mu_\nu - S^\mu_{\ \nu}\,, \tag{172}$$

but the reduces in the Newtonian limit (151) simply to $y^\mu_\nu = \gamma^\mu_\nu$.

By substitution of the kinematic formula (170) into the dynamical condition (171) we finally obtain the required characteristic equation in the form

$$(\upsilon^2 \rho\, y^{\mu\nu} - Q^{\mu\nu})\hat{u}_\nu = 0\,. \tag{173}$$

This is an effectively 3-dimensional eigenvector equation with υ^2, the square of the relative propagation speed, as eigenvalue, for which the eigenvector is the covariant velocity gradient discontinuity amplitude $\hat{u}_\mu$ as characterised by the orthogonality condition

$$u^\mu \hat{u}_\mu = 0\,. \tag{174}$$

In terms of the Hadamard elasticity tensor specified according to (163), the characterists matrix $Q^{\mu\nu}$ can be seen to be expressible as a function of the propagation direction, as indicated by the spacelike unit covector ν_μ by the formula

$$Q^{\mu\nu} = A^{\mu\rho\nu\sigma}\nu_\rho \nu_\sigma\,. \tag{175}$$

The simplest application of this formula is of course to the case of a medium that is intrinsically isotropic (as will typically be the case in macroscopic applications involving averaging over a large number of randomly oriented mesoscopic crystals) and that is in an undeformed, though perhaps highly compressed state, with energy density ϵ_0 say. In such an undeformed state the stress tensor $S^{\mu\nu}$ will reduce to an undeformed value, $S_0^{\mu\nu}$ say, that will be characterised just by a pressure scalar P_0, in terms of which it will take the form

$$S_0^{\mu\nu} = -P_0\gamma^{\mu\nu}\,. \tag{176}$$

It follows that the tensor $y^{\mu\nu}$ in (173) will reduce to an undeformed value $y_0^{\mu\nu}$ given in the relativistic case by

$$y_0^{\mu\nu} = \left(1 + \frac{\epsilon_0 + P_0}{\rho\, c^2}\right)\gamma^{\mu\nu}\,. \tag{177}$$

As discussed in the cited work [21] on the relativistic case, for such a state the elasticity tensor $E^{\mu\nu\rho\sigma}$ will reduce to a corresponding isotropic value $E_0^{\mu\nu\rho\sigma}$ that is expressible in the well known form

$$E_0^{\mu\nu\rho\sigma} = \left(\beta_0 - \tfrac{1}{3}P_0\right)\gamma^{\mu\nu}\gamma^{\rho\sigma} + 2(\mu_0 + P_0)\left(\gamma^{\mu(\rho}\gamma^{\sigma)\nu} - \tfrac{1}{3}\gamma^{\mu\nu}\gamma^{\rho\sigma}\right), \quad (178)$$

in which the coefficients β_0 and μ_0 are respectively interpretable as the bulk modulus and the modulus of rigidity. In the particular case of a perfect fluid the rigidity will vanish, $\mu = 0$, and the bulk modulus will be given by the derivative of the pressure with respect to fractional volume change, $\beta = n\mathrm{d}P/\mathrm{d}n$. According to (163) the Hadamard elasticity tensor $A^{\mu\nu\rho\sigma}$ reduce to a corresponding isotropic limit value given by

$$A_0^{\mu\nu\rho\sigma} = \beta_0\gamma^{\mu\nu}\gamma^{\rho\sigma} + 2P_0\gamma^{\mu[\sigma}\gamma^{\nu]\rho} + 2\mu_0\left(\gamma^{\mu(\rho}\gamma^{\sigma)\nu} - \tfrac{1}{3}\gamma^{\mu\nu}\gamma^{\rho\sigma}\right), \quad (179)$$

which is such that the antisymmetric pressure term will cancel out in the formula for the characteristic matrix $Q^{\mu\nu}$, leaving an expression of the same form $Q_0^{\mu\nu}$ as is familiar in the low pressure limit, namely

$$Q_0^{\mu\rho} = \left(\beta_0 + \tfrac{1}{3}\mu_0\right)\nu^\mu\nu^\rho + \mu_0\gamma^{\mu\rho}. \quad (180)$$

9 Faraday – Ampere magnetodynamics

So long as it acts merely as a given prescribed background, an electromagnetic field can be treated within a Newtonian framework in a manner that is satisfactorily Galilei and even Milne invariant [26]. However when it is necessary to treat it in its own right as an active dynamical field governed by an unrestricted electric current source j^μ then it is necessary to sacrifice Galilean (and hence a fortiori Milne) invariance, which was brutally violated by the introduction of a physically preferred ether in Maxwell's original formulation, but more elegantly replaced by Lorentz invariance in Einstein's special relativistic treatment.

A satisfactorily Galilean and even Milne invariant formulation in terms of an antisymmetric field 2-form with components $F_{\mu\nu} = -F_{\nu\mu}$ can however be set up as a self consistent approximation in cases for which only a subset of three "magnetic" degrees of freedom are dynamically independent, while the other three "electric" components (out of the six contained in $F_{\mu\nu}$) and all the components of the current are treated merely as passively derived fields in the manner that will be described immediately below in this section.

In the following section it will be shown how such an ether gauge invariant Faraday-Ampere type model can be coupled, in a variational formulation, to a simple perfect barotropic fluid or elastic solid model, of the kind described in the previous section, in the special case of "perfect conductivity", meaning the case for which the field is "purely magnetic" in the sense that, with respect to the local rest frame specified by the velocity 4 vector u^μ of the fluid, the relevant electric components are zero. We thus obtain a 4-dimensionally covariant formulation of the non-relativistic version of what is known in the fluid case [27] as a perfect magnetohydrodynamics.

In the generic case, the electric and magnetic fields E_μ and $B_{\mu\nu}$ can be defined, with respect to an ether vector e^μ, by the decomposition

$$F_{\mu\nu} = B_{\mu\nu} + 2E_{[\mu}t_{\nu]}\,, \tag{181}$$

subject to the conditions

$$E_\mu e^\mu = 0\,, \quad B_{\mu\nu}e^\nu = 0\,, \tag{182}$$

which are equivalent to the specification

$$E_\mu = F_{\mu\nu}e^\nu\,, \tag{183}$$

where, in the Newtonian case we are concerned with here, t_μ is the preferred time covector introduced in (1) (while in the relativistic case it would be given in terms of the spacetime metric by $t_\mu = -g_{\mu\nu}u^\mu$). Under the action of an ether gauge transformation of the form (12), as generated by a spacelike boost vector field b^μ, these fields will acquire new values given by

$$\breve{E}_\mu = E_\mu + (B_{\mu\nu} - t_\mu E_\nu)b^\nu\,, \quad \breve{B}_{\mu\nu} = B_{\mu\nu} + 2t_{[\mu}B_{\nu]\rho}b^\rho\,. \tag{184}$$

but the gauge dependence of the corresponding contravectorial quantities

$$E^\mu = \gamma^{\mu\nu}E_\nu\,, \quad B^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho}B_{\nu\rho}\,, \tag{185}$$

will be simpler, so much so that the vector B^μ will actually be physically well defined in the sense of being independent of the choice of ether frame, since we shall have

$$\breve{E}^\mu = E^\mu + \gamma^{\mu\nu}B_{\nu\rho}b^\rho\,, \quad \breve{B}^\mu = B^\mu\,. \tag{186}$$

The kinematic field 2-form closure condition

$$\nabla_{[\mu}F_{\nu\rho]} = 0\,, \tag{187}$$

corresponds two of the 4-Maxwell equations, which are expressible in our 4-dimensionally covariant notation scheme as

$$\nabla_\mu B^\mu = 0\,, \quad \epsilon^{\mu\nu\rho}\nabla_\nu E_\rho = -e^\nu\nabla_\nu B^\mu\,, \tag{188}$$

of which the second is interpretable as Faraday's law of magnetic induction.

The trouble, in a Newtonian framework, is with the other two Maxwell equations, which specify the way an arbitrary source current 4-vector j^μ governs the dynamic evolution of the field. In a relativistic theory this is done by setting $\nabla_\nu F^{\mu\nu} = 4\pi j^\mu$, where $F^{\mu\nu}$ is obtained from $F_{\mu\nu}$ by contraction with the non-degenerate contravariant spacetime metric $g^{\mu\nu}$. However the analogous Newtonian procedure of contraction with $\gamma^{\mu\nu}$ will, due to the degeneracy of the latter, give a result that is overdetermined, having a form that is expressible in terms of the rationalised magnetic field

$$H^{\mu\nu} = \frac{1}{4\pi}\gamma^{\mu\rho}\gamma^{\nu\sigma}F_{\rho\sigma} = \frac{1}{4\pi}\gamma^{\mu\rho}\gamma^{\nu\sigma}B_{\rho\sigma}\,, \tag{189}$$

which is ether gauge independent, and purely spacelike,

$$\breve{H}^{\mu\nu} = H^{\mu\nu} \quad H^{\mu\nu} t_\nu = 0\,, \tag{190}$$

as the Ampere type equation

$$\nabla_\nu H^{\mu\nu} = j^\mu\,. \tag{191}$$

It is evident from (190) that this can be satisfied only if the current is restricted to be similarly spacelike, in the sense of satisfying the consistency condition

$$j^\mu t_\mu = 0\,. \tag{192}$$

The associated electromagnetic action density $\Lambda_{\rm F}$, as similarly obtained from the usual relativistic action density $F^{\mu\nu} F_{\nu\mu}/16\pi$ by substituting $\gamma^{\mu\nu}$ for $g^{\mu\nu}$, will have the form

$$\Lambda_{\rm F} = -\epsilon_{\rm F}\,, \tag{193}$$

where $\epsilon_{\rm F}$ is the ether gauge independent magnetic energy density as given by

$$\epsilon_{\rm F} = \frac{B^2}{8\pi}\,, \qquad B^2 = \eta_{\mu\nu} B^\mu B^\nu = \tfrac{1}{2}\gamma^{\mu\rho}\gamma^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\,, \tag{194}$$

from which, by considering the effect of varying the degenerate metric $\gamma^{\mu\nu}$ for fixed $F_{\mu\nu}$, the corresponding stress energy tensor is obtainable according to the ansatz (72) in the form

$$T^{\ \mu}_{{\rm F}\ \nu} = H^{\mu\rho} F_{\nu\rho} - \epsilon_{\rm F} \delta^\mu_\nu\,. \tag{195}$$

The force density acting on the field (the opposite of the Faraday - Lorentz type electromagnetic reaction on the relevant medium) will therefore be given by the expression

$$f_{{\rm F}\nu} = \nabla_\mu T^{\ \mu}_{{\rm F}\ \nu} = j^\mu F_{\mu\nu}\,. \tag{196}$$

This necessary restriction (192) is interpretable as meaning that there can be no net electric charge density, a condition which replaces the traditional Coulomb equation that would be expressed in our covariant notation scheme as $\nabla_\nu E^\nu = 4\pi j^\mu t_\mu$, but which can be seen from (186) to be incompatible with Galilean invariance unless the magnetic part of the field is absent, and which even then will be incompatible with (184) except in the pure vacuum case for which there is no source current j^μ at all.

To set up an ether frame invariant model for use as an exactly self consistent approximation in a Newtonian framework we are thus faced with a choice between two generically incompatible alternatives. One possibility (which is likely to be most realistic when insulating material is involved) is to use a scheme based on the Coulomb equation, which entails abandoning the Ampere equation and simply restricting the magnetic part of the field to be zero. The other possibility (more likely to be realistic for dealing with good conductors) which is the option chosen for the present work, is to use a scheme based on the Ampere equation (191) in conjunction with the force law (196), which entails abandoning the Coulomb equation and restricting the charge density to be zero in accordance with (190). This effectively demotes the current from the status of an independent dynamical variable to that of a derived quantity, and entails a concomitant loss of independence of the electric part of the field, which instead of the Coulomb equation, is required, in the most commonly used kind of model, to obey an Ohm type equation, of which the simplest version is covariantly expressible, in terms of the 4-velocity vector u^μ of the relevant supporting – fluid or solid – medium, as $\gamma^{\mu\nu}F_{\mu\nu}u^\nu = \kappa j^\mu$, where κ is a resistivity scalar that, in the case of a non-isotropic solid, might need to be replaced by a tensor. For positive resistivity, $\kappa > 0$, such an Ohm ansatz can be applied in the case of a composite medium involving entropy density as an independent degree of freedom, but its substitution in the force law (196) shows that it will entail a generically positive rate of energy transfer to the medium that will be given as a quadratic function of the (purely spacelike) current (191) by an expression of the form $u^\nu f_{\mathrm{F}\nu} = \kappa j^2$, where $j^2 = \eta_{\mu\nu} j^\mu j^\nu$. It is evident however that this will not in general be compatible with the identity (148) that must be satisfied for a single constituent medium of the kind to which the present study is restricted. To obtain a self consistent model involving just a simple solid or (barotropic) fluid, we need to restrict ourselves to the non-dissipative perfectly conducting case case for which the resistivity vanishes, $\kappa = 0$.

10 Perfect magneto – elastic dynamics

It is evident from the foregoing considerations that the perfect conductivity condition needed to characterise a medium of the simple non dissipative kind considered here reduces to the perfect conductivity condition that is expressible covariantly as the condition

$$F_{\mu\nu}u^\mu = 0\,, \tag{197}$$

which is interpretable as meaning that with respect to the local rest frame specified by the material 4-velocity u^μ the field is of a purely magnetic character. This condition is not just mathematically convenient but also justifiable – due to the relatively small mass of the electrons that are typically the main charge carriers – as a remarkably good approximation in many terrestrial applications and in a very wide range of astrophysical contexts, of which the most extensively studied so far have been those for which the the relevant material medium is a simple perfect fluid, in which case the ensuing theory is what is known as perfect magnetohydrodynamics. As has been shown by the work of Jacob Bekenstein with Eleizer and Asaf Oron [27, 28], the elegant mathematical properties of this kind of magnetohydrodynamic model are easier to analyse in a fully relativistic framework. Part of the motivation for the 4-dimensionally covariant approach developed here is to facilitate the extra work needed [27] for the treatment of the Newtonian limit. It is to be noted that the variational formulation developed below differs, in the fluid limit, from the one developed by Bekenstein and Oron [27] who worked with Clebsch type potentials of the kind introduced in a relativistic context by Schutz [17]. The use of such Clebsch type variables is just one of several possibilities that may be convenient for various purposes in a purely fluid context, but like most of the other alternatives it has the disadvantage of being unsuitable for generalisation to solids. For the purpose of setting up a variational formulation for the treatment of an elastic solid medium it has long been clear [15] that the only practical option is to work in terms of world line displacements as characterised by comoving coordinate variable of the kind denoted here by q^A .

As a consequence of the closure property (187), it follows that the 2 - form $F_{\mu\nu}$ will be "frozen in" in the sense of having vanishing Lie derivative with respect to the flow:

$$\vec{u}\pounds\, F_{\mu\nu} \equiv u^\rho \nabla_\rho F_{\mu\nu} + 2F_{\rho[\nu}\nabla_{\mu]}u^\rho = 0\,. \tag{198}$$

since an antisymmetric matrix cannot have even rank, the orthogonality condition (197) implies that, as well as u^μ, the field $F_{\mu\nu}$ possesses another independent null eigenvector, which can be taken to be B^μ as given by (185). A well known consequence is that the Maxwellian 2-form $F_{\mu\nu}$ will be conserved by Lie transport along any vector that is a linear combination of the form $\xi^\mu = \alpha_1 B^\mu + \alpha_2 u^\mu$ where α_1 and α_2 are any scalar fields, and it can also be seen [16] that the 2-surface elements spanned by such vectors will mesh together to form a congruence of well defined flux 2-surfaces.

In the same way as remarked above about the stress tensor $S_{\mu\nu}$, the world line orthogonality property (197) is interpretable as meaning that $F_{\mu\nu}$ naturally determines and is determined by a corresponding antisymmetric material base tensor with components $F_{_{AB}}$ such that

$$F_{\mu\nu} = F_{_{AB}} q^{_A}_{\ ,\mu} q^{_B}_{\ ,\nu}\,, \tag{199}$$

while the Lie transport condition (198) is interpretable as meaning that this induced field will be time independent,

$$\dot{F}_{_{AB}} = 0 \tag{200}$$

so that the covariant components $F_{_{AB}}$ will be those of a fixed 2-form field on the 3 dimensional base manifold $\mathcal{X}$. Morever it can be seen that as a consequence of the space-time closure property (187) the base space 2-form field will have a corresponding closure property,

$$F_{[_{AB},_{C}]} = 0\,, \tag{201}$$

which means that it will locally be expressible as the exterior derivative,

$$F_{_{AB}} = 2A_{[_B,_A]}\,, \tag{202}$$

of a "frozen in" 1-form field with components $A_{_A}$ of the kind introduced in (90), whose spacetime pull back (88) thereby provides an expression of the familiar form

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}\,. \tag{203}$$

It is to be remarked that this natural material gauge is not uniquely defined, since there is still some liberty in the choince of $A_{_B}$. As a result of the degeneracy property (197), there exists a current η^μ defined by

$$\eta^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}\,, \tag{204}$$

that is conserved in the sense of satisfying

$$\nabla_\mu \eta^\mu = \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 0, \tag{205}$$

and that has a time component

$$\eta^\mu t_\mu = -A_\mu B^\mu \tag{206}$$

which is proportional to the well known magnetic helicity scalar [29].

As a consequence of this "frozen in" behaviour, it can be seen that allowance for the effect of the magnetic field can be included directly within the perfect elasticity formalism developed in Section 5 simply by taking the energy density to have the form

$$\epsilon = \epsilon_0 + \epsilon_{\rm F} \tag{207}$$

in which ϵ is the purely material part – depending just on the base coordinates q^A and the induced metric components γ^{AB} – that would remain when the field components F_{AB} components are set to zero, while the other part $\epsilon_{\rm F}$ is specified as a function not just of q^A and γ^{AB} but also of the field components F_{AB} whose status will be that of initial data that – subject to the closure condition (201) are freely specifiable, but that once chosen will evolve as fixed functions of the base coordinates q^A. For a fixed base value of the base coordinates q^A the most general variation of ϵ will determine not just a corresponding (symmetric) stress tensor S^{AB} but also a corresponding (antisymmetric) magnetic field tensor by the prescription

$$\delta\epsilon = \tfrac{1}{2}(S^{AB} - \epsilon\gamma^{AB})\delta\gamma_{AB} + \tfrac{1}{2}H^{AB}\delta F_{AB}, \tag{208}$$

which decomposes with

$$S^{AB} = S_0^{AB} + S_{\rm F}^{AB}, \tag{209}$$

as the sum of parts given by

$$\delta\epsilon_0 = \tfrac{1}{2}\big(S_0^{AB} - \epsilon_0\gamma_{AB}\big)\delta\gamma_{AB}, \tag{210}$$

and

$$\delta\epsilon_{\rm F} = \tfrac{1}{2}\big(S_{\rm F}^{AB} - \epsilon_{\rm F}\gamma^{AB}\big)\delta\gamma_{AB} + \tfrac{1}{2}H^{AB}\delta F_{AB}\,. \tag{211}$$

In the general case of a polarised medium the functional form of $\epsilon_{\rm F}$ might be rather elaborate, but in the simple case of an unpolarised medium it is simply to be identified with the ordinary vacuum magnetic energy density as given by (194) which is translatable into terms material base fields as

$$\epsilon_{\rm F} = \tfrac{1}{4}H^{AB}F_{AB}\,, \tag{212}$$

where, consistently with (211) the magnetic field tensor H^{AB} is given in this particular case simply by

$$H^{AB} = \frac{1}{4\pi}\gamma^{AC}\gamma^{BD}F_{CD}\,. \tag{213}$$

while the corresponding magnetic stress contribution will be obtainable from (211) as

$$S_{{\rm F}\,B}^{\,A} = H^{AC}F_{CB} + \epsilon_{\rm F}\gamma_B^A\,. \tag{214}$$

which is equivalent to the expression given, using the notation (185), as

$$S_{\rm F}^{\mu\nu} = \frac{1}{8\pi}(2B^\mu B^\nu - B^2\gamma^{\mu\nu}). \tag{215}$$

In order to consider the effect on wave propagation, as discussed in Section 8, we need the ensuing replacement of (177) for the tensor defined in the relativistic case by (172), which works out as

$$y^{\mu\nu} = \left(1 + \frac{\epsilon_0 + P_0}{\rho\, c^2}\right)\gamma^{\mu\nu} + \frac{1}{4\pi c^2 \rho}(B^2\gamma^{\mu\nu} - B^\mu B^\nu)\,. \tag{216}$$

We also need to evaluate the corresponding (unpolarised) magnetic elasticity contribution. This will be obtainable on the basis of the ansatz (162), which provides the expression

$$A_{\rm F}^{ACBD} = E_{\rm F}^{ACBD} + \gamma^{AB} S_{\rm F}^{CD} = H^{AB}B^{CD} + H^{DA}B^{BC} + \epsilon_{\rm F}\big(\gamma^{AB}\gamma^{CD} + 2\gamma^{A[D}\gamma^{C]B}\big)$$

$$+2S_{\rm F}^{A[C}\gamma^{D]B} + 2S_{\rm F}^{B[C}\gamma^{D]A} - S_{\rm F}^{AB}\gamma^{CD}\,. \tag{217}$$

It immediately follows that the corresponding magnetic contribution in the characteristic matrix $Q^{\mu\nu}$ given by (175) will be expressible in terms of the relevant magnetic field tensor as

$$Q_{\rm F}^{\mu\nu} = A_{\rm F}^{\mu\rho\nu\sigma}\nu_\rho\nu_\sigma = -4\pi\big(H^{\mu\rho}H_\rho^{\ \nu} + H^{\mu\rho}H^{\nu\sigma}\nu_\rho\nu_\sigma\big)\,. \tag{218}$$

By introducing a spacelike unit vector ι^μ that is chosen in the polarisation plane orthogonal to ν^μ (the unit vector in the direction of polarisation) in such a way that the magnetic induction vector will be expressible as

$$B^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = B_{\|}\nu^\mu + B_\perp\iota^\mu\,, \tag{219}$$

we can rewrite (218) as

$$4\pi\, Q_{\rm F}^{\mu\nu} = B_\perp^2\nu^\mu\nu^\nu - 2B_{\|}B_\perp\nu^{(\mu}\iota^{\nu)} + B_{\|}^2(\gamma^{\mu\nu} - \nu^\mu\nu^\nu)\,. \tag{220}$$

The complete characteristic matrix,

$$Q^{\mu\nu} = Q_0^{\mu\nu} + Q_{\rm F}^{\mu\nu} \tag{221}$$

can easily be obtained in an explicit form if we suppose that the material contribution $Q_0^{\mu\nu}$ has the form (180) that is relevant when the medium is in a simple isotropic state of the kind characterised by (178). In this (unpolarised, materially isotropic) case, we obtain an expression of the form

$$Q^{\mu\nu} = Q_{\|}\nu^\mu\nu^\nu - 2Q_\times\nu^{(\mu}\iota^{\nu)} + Q_\perp(\gamma^{\mu\nu} - \nu^\mu\nu^\nu)\,. \tag{222}$$

with

$$Q_{\|} = \beta_0 + \frac{4\mu_0}{3} + \frac{B_\perp^2}{4\pi}\,,\qquad Q_\times = \frac{B_{\|}B_\perp}{4\pi}\,,\qquad Q_\perp = \mu_0 + \frac{B_{\|}^2}{4\pi}\,. \tag{223}$$

The characteristic eigenvector equation (173) is easily soluble in the non-relativistic case, for which we simply have $y^{\mu\nu} = \gamma^{\mu\nu}$: it can be seen that there will always be a transverse Alfven type mode, with polarisation covector $\hat{u}_\mu$ orthogonal to both the propagation direction and the magnetic field direction, whose velocity $\upsilon_\top$ will be given by

$$\upsilon_\top^2 = \frac{Q_\perp}{\rho} = \frac{\mu_0}{\rho} + \frac{B_\parallel^2}{4\pi\rho}\,, \tag{224}$$

an expression that reduces to the well known formula $\upsilon_\top^2 = \mu_0/\rho$ for propagation of transverse ("wiggle" or "shake") modes in an isotropic solid when the magnetic field is absent.

In the case of propagation parallel to the magnetic field direction, meaning $\nu^\mu \propto B^\mu$ so that $B_\perp = 0$, there will be a second (orthogonally polarised) transverse mode, with the same propagation speed $\upsilon_\top$, which in this case will be given by the expression $\upsilon_\top^2 = \mu_0/\rho + B^2/4\pi\rho$, which reduces to the well known Alfven formula $\upsilon_\top^2 = B^2/4\pi\rho$ in the magnetohydrodynamic (purely fluid) limit in which the rigdity coefficient μ vanishes. There will also be a purely longitudinal (sound type) mode with velocity $\upsilon_\parallel$ that will be given, independently of the magnetic field strength, by the expression

$$\upsilon_\parallel^2 = \frac{\beta_0}{\rho} + \frac{4\mu_0}{3\rho}\,, \tag{225}$$

which reduces in the magnetohydrodynamic limit, $\mu \to 0$, just to Newton's formula $\upsilon_\parallel^2 = \mathrm{d}P/\mathrm{d}\rho$ for the speed of ordinary sound.

More generally, when the propagation is not parallel to the magnetic field, the other two modes will be of mixed – partially longitudinal, partially transverse – type with polarisation in the plane generated by the propagation direction and the magnetic field direction, and with speeds υ_+ and υ_- that will be obtainable as the roots of the eigenvalue equation

$$(Q_\parallel - \rho\upsilon^2)(Q_\perp - \rho\upsilon^2) - Q_\times^2 = 0\,, \tag{226}$$

which gives

$$2\rho\,\upsilon_\pm^2 = Q_\parallel + Q_\perp \pm \sqrt{(Q_| - Q_\perp)^2 + 4Q_\times^2}\,. \tag{227}$$

These solutions can be seen to be such that we shall have $\upsilon_+ \to \upsilon_\parallel$ and $\upsilon_- \to \upsilon_\top$ in the limit of parallel propagation for which $B_\perp \to 0$ and $B_\parallel \to B$.

In the relativistic case we shall still be able to use the same expressions (222) and (223) for $Q^{\mu\nu}$, but for the tensor $y^{\mu\nu}$ in the characteristic equation (173) it will be necessary to use the less simple formula (216), which will be expressible, in a form analogous to (222), as

$$\rho\,c^2 y^{\mu\nu} = Y_\parallel \nu^\mu\nu^\nu - 2Q_\times \nu^{(\mu}\iota^{\nu)} + Y_\perp \iota^\mu\iota^\nu + Y(\gamma^{\mu\nu} - \nu^\mu\nu^\nu - \iota^\mu\iota^\nu)\,. \tag{228}$$

with

$$Y_\parallel = c^2\tilde{\rho}_0 + \frac{B_\perp^2}{4\pi}\,,\quad Y_\perp = c^2\tilde{\rho}_0 + \frac{B_\parallel^2}{4\pi}\,,\quad Y = c^2\tilde{\rho}_0 + \frac{B^2}{4\pi}\,, \tag{229}$$

where

$$c^2\tilde{\rho}_0 = c^2\rho + \epsilon_0 + P_0\,. \tag{230}$$

As before, there will always be a transverse Alfven type mode, with polarisation covector $\hat{u}_\mu$ orthogonal to both the propagation direction and the the magnetic field direction, with velocity $\upsilon_{\top}$ that will be given generically by

$$\frac{\upsilon_{\top}^2}{c^2} = \frac{Q_\perp}{Y} = \frac{4\pi\mu_0 + B_\parallel^2}{4\pi c^2\tilde{\rho}_0 + B^2}\,. \tag{231}$$

In the case of propagation in the direction of the magnetic field, i.e. when $B_\perp = 0$ there will again be a second (orthogonally polarised) transverse mode, with the same propagation speed $\upsilon_{\top}$, as well as a purely longitudinal (sound type) mode with velocity $\upsilon_\parallel$ that will be given in the generic case by the same formula as has long been well known [15] for the unmagnetised case, namely

$$\upsilon_\parallel^2 = \frac{\beta}{\tilde{\rho}_0} + \frac{4\mu_0}{3\tilde{\rho}_0}\,. \tag{232}$$

For the generic case, in which the propagation is not parallel to the magnetic field, the relativistic generalisation of the equation (227) for the speeds υ_+ and υ_- of the mixed modes – with polarisation in the plane generated by the propagation direction and the magnetic field direction – will be expressible in terms of the dimensionless ratios

$$\mathcal{Q}_\parallel = \frac{Q_\parallel}{Y_\parallel}\,, \quad \mathcal{Q}_\perp = \frac{Q_\perp}{Y_\perp}\,, \quad \mathcal{Q}_\times = \frac{Q_\times}{\sqrt{Y_\parallel Y_\perp}}\,, \tag{233}$$

by

$$\frac{\upsilon_\pm^2}{c^2} = \frac{\mathcal{Q}_\parallel + \mathcal{Q}_\perp - 2\mathcal{Q}_\times^2 \pm \sqrt{(\mathcal{Q}_\parallel - \mathcal{Q}_\perp)^2 + 4\mathcal{Q}_\times^2(1-\mathcal{Q}_\parallel)(1-\mathcal{Q}_\perp)}}{2(1-\mathcal{Q}_\times^2)}\,, \tag{234}$$

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