Superfluidity and Superconductivity in Compact Stars

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NewCompStar School, Bucharest, 20-25 September 2015

Outline

1. Superfluidity and superconductivity in the laboratory
   - Basic phenomenology and historical context
   - Phenomenological theories
   - Microscopic theories

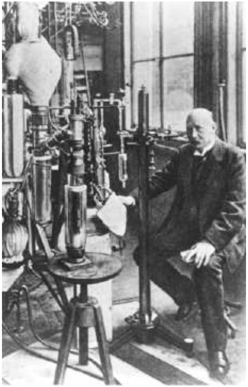
2. Superfluidity and superconductivity in neutron stars
   - Theories of nuclear superfluidity and superconductivity (quark pairing will be addressed by Prof. Armen Sedrakian)
   - Observational evidence

Disclaimer: these lectures will not cover all aspects of superfluidity and superconductivity, but will focus on those most relevant to a basic understanding of these phenomena in neutron stars.

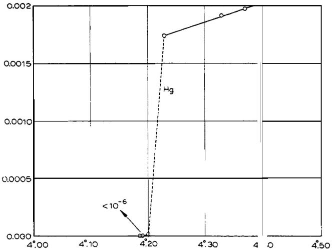
Part 1: Superfluidity and superconductivity in the laboratory

Discovery of "supraconductivity"

Heike Kamerlingh Onnes and his collaborators were the first to liquefy helium in 1908.



On April 8th, 1911, H. K. Onnes and Gilles Holst discovered that the electric resistance of mercury dropped to almost zero at $T_c \simeq 4.2$ K



Onnes was awarded the Nobel Prize in 1913.

Two years later, lead was found to be also superconducting. Other superconducting elements and metallic compounds were discovered in the following decades.

Persistent electric currents

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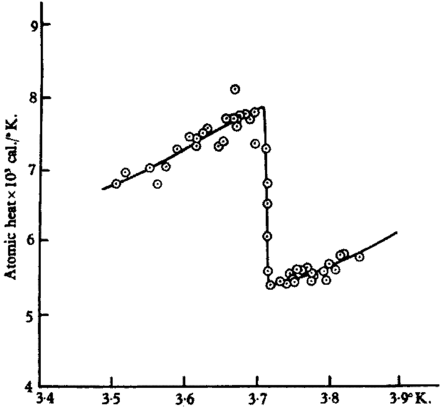
In superconducting rings, the decay time of induced electric currents is not less than 100 000 years !
J. File and R. G. Mills, PRL 10, 93 (1963)

Heat capacity

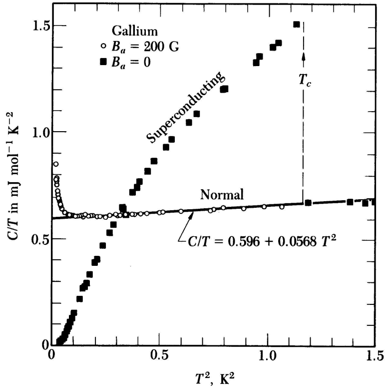
In 1932, Keesom and Kok found that the heat capacity of tin exhibits a discontinuity at T_c thus showing that the superconducting transition is of second order.

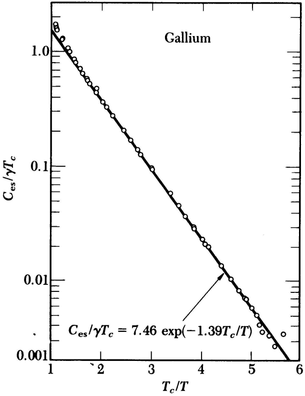


Keesom and Kok, Proc. Roy. Acad. Amsterdam 35, 743 (1932).

Heat capacity

At temperatures $T < T_c$ the electron heat capacity is exponentially suppressed suggesting the existence of an **gap in the electron energy spectrum**.

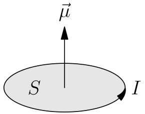




Kittel, Introduction to Solid State Physics

Intermission: magnetostatics in a magnetic material

In a magnetic material, the set of microscopic magnetic dipole moments $\boldsymbol{\mu}$ give rise to a **magnetization current** $\boldsymbol{j}_m = c\nabla \times \boldsymbol{M}$ (cgs), where the **magnetization** $\boldsymbol{M}$ is the macroscopic density of magnetic moments.



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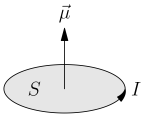

Introducing the **auxiliary magnetic field $\boldsymbol{H} \equiv \boldsymbol{B} - 4\pi\boldsymbol{M}$** (to avoid confusion $\boldsymbol{B}$ is usually referred to as the **magnetic induction**), Maxwell-Ampere's equation can be expressed as

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}_{\mathrm{free}}$$

with $\boldsymbol{j}_{\mathrm{free}} = \boldsymbol{j} - \boldsymbol{j}_m$ is the "free" electric current associated with the applied field.

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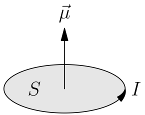

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with $\boldsymbol{j}_{\text{free}} = \boldsymbol{j} - \boldsymbol{j}_m$ is the "free" electric current associated with the applied field.

Note however that $\boldsymbol{H}$ is not uniquely determined by $\boldsymbol{j}_{\text{free}}$ since

$$\nabla \cdot \boldsymbol{H} = -4\pi\nabla \cdot \boldsymbol{M}$$

Therefore $\boldsymbol{H}$ does not generally coincide with the applied field.

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In isotropic materials, $\boldsymbol{H}_d = -\frac{\gamma}{4\pi}\boldsymbol{M}$ where the demagnetizing factor γ depends on the geometry of the material. In particular, $\gamma = 0$ for a long thin sample and a magnetic field applied along the symmetry axis.

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where χ is the **magnetic susceptibility** of the material. In such case, we have

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A material is

- **paramagnetic** if $\chi > 0$ under an applied field,
- **diamagnetic** if $\chi < 0$ under an applied field.

Typically $|\chi_{\text{diamagnetic}}| \sim 10^{-5} \ll \chi_{\text{paramagnetic}}$.

Some (e.g. ferromagnetic) materials may have a permanent magnetization, i.e. $\chi \neq 0$ even in the absence of an applied field.

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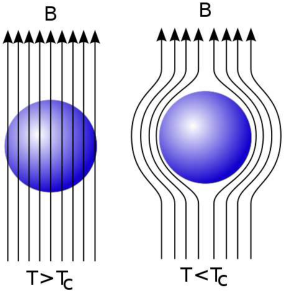

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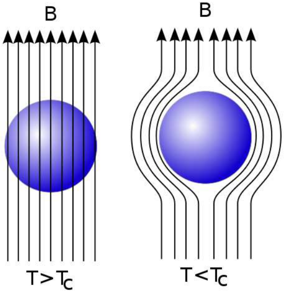

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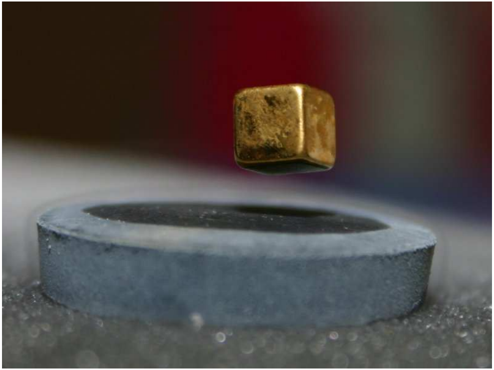
This phenomenon showed that a superconductor is not just a perfect conductor but correspond to a new thermodynamic state of matter.

Indeed, Ohm's law $\boldsymbol{j} = \sigma \boldsymbol{E}$ implies $\boldsymbol{E} = 0$ if $\sigma \to +\infty$. From Maxwell Faraday equation, $\dfrac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} = 0$: $\boldsymbol{B}$ should not change.

Magnetic levitation

As a spectacular consequence of the Meissner-Ochsenfeld effect, a magnet can be levitated over a superconducting material.



http://www.mn.uio.no/fysikk/english/research/groups/amks/superconductivity/levitation/

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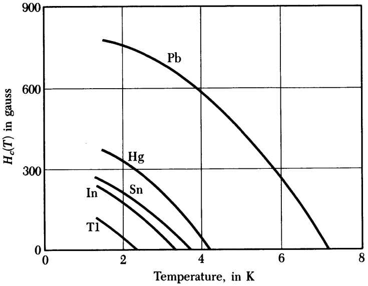

Experimentally, it is found that

$$H_c(T) = H_0\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

Kittel, Introduction to Solid State Physics

The critical magnetic field implies the existence of a **critical electric current**.

Thermodynamics of a superconductor

The thermodynamic state of a superconductor at a given temperature T and magnetic intensity H is determined by the **generalized free energy**

$$F(T, H) = U - TS - \frac{1}{4\pi}\boldsymbol{H} \cdot \boldsymbol{B}.$$

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$$F_S(T,H) - F_N(T,H) = \frac{1}{8\pi}\left(H^2 - H_c(T)^2\right) \leq 0 \text{ since } H \leq H_c(T)$$

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Noting that $S = -\left.\frac{\partial F}{\partial T}\right|_H$, we obtain for the latent heat of the transition

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Experimentally, C_S is exponentially suppressed therefore $\gamma = \frac{H_0^2}{4\pi T_c^2}$

$$\left.\frac{C_S}{C_N}\right|_{T=T_c} = 3$$

London theory

In 1935, Fritz and Heinz London proposed the following constitutive equation for a (simply connected) superconductor: $\boldsymbol{j} = -\frac{nq^2}{m}\boldsymbol{A}$, where n is the number density of superconducting particles, q their electric charge, and m their mass.

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Taking the curl of Maxwell-Ampere's equation $\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}$,

$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2\boldsymbol{B} = -\nabla^2\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{\nabla} \times \boldsymbol{j}$, and using London's equation $\boldsymbol{j} = -\frac{nq^2}{m}\boldsymbol{A}$, we obtain

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Remarks

- The only solution corresponding to a uniform field inside the superconductor is $\boldsymbol{B} = 0$.
- London's equation implies the following gauge: $\nabla \cdot \boldsymbol{A} = 0$, and $\boldsymbol{A}_\perp = 0$ at the surface of the superconductor since $\boldsymbol{j}_\perp = 0$.

Application: semi-infinite superconductor

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The electron current is mainly located in the surface since $j_y(x) = -\frac{c}{4\pi}\frac{dB}{dx} = \frac{cB(0)}{4\pi\lambda_L} \exp(-x/\lambda_L)$, and $j_x = j_z = 0$.

Note that in thin films with thickness $d \ll \lambda_L$, the Meissner effect is not complete therefore the thermodynamic approach breaks down. The critical field H_c parallel to the film is very high.

Pippard theory

Pippard proposed to extend London theory so as to allow for **non-local effects**:

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{3nq^2}{4\pi m\xi_0}\int d^3r' \, \frac{\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{A}(\boldsymbol{r'}))}{R^4}\exp\left(-\frac{R}{\xi}\right)$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$, and ξ represents a **coherence length**.

In the presence of impurities, $\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{\ell}$ where ℓ is the mean-free path and ξ_0 is the coherence length of a pure sample. London's theory corresponds to the limit $\ell \to +\infty$ and $\xi_0 \to 0$:

$$j_i(\boldsymbol{r}) \approx -\frac{3nq^2}{4\pi m\xi_0}A_j(\boldsymbol{r})\int d^3r' \, \frac{R_iR_j}{R^4}\exp\left(-\frac{R}{\xi_0}\right)$$

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In the presence of impurities, $\dfrac{1}{\xi} = \dfrac{1}{\xi_0} + \dfrac{1}{\ell}$ where ℓ is the mean-free path and ξ_0 is the coherence length of a pure sample. London's theory corresponds to the limit $\ell \to +\infty$ and $\xi_0 \to 0$:

$$j_i(\boldsymbol{r}) \approx -\frac{3nq^2}{4\pi m\xi_0} A_j(\boldsymbol{r}) \int d^3R \, \frac{1}{3}\delta_{ij} \frac{\exp\left(-R/\xi_0\right)}{R^2}$$

Pippard theory

Pippard proposed to extend London theory so as to allow for **non-local effects**:

$$\boldsymbol{j}(\boldsymbol{r}) = -\frac{3nq^2}{4\pi m\xi_0}\int d^3r'\,\frac{\boldsymbol{R}(\boldsymbol{R}\cdot\boldsymbol{A}(\boldsymbol{r}'))}{R^4}\exp\left(-\frac{R}{\xi}\right)$$

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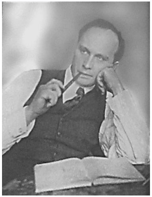
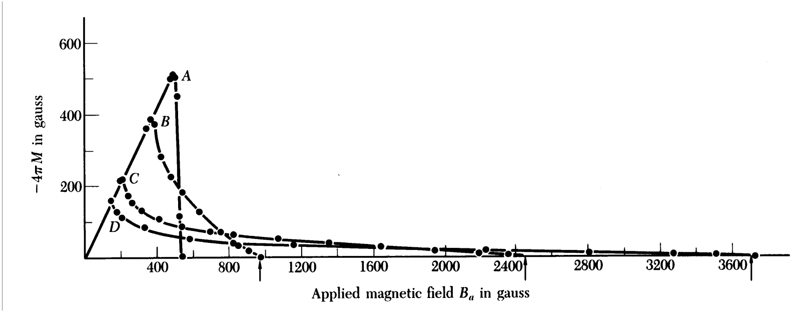
$$\boldsymbol{j}(\boldsymbol{r}) \approx -\frac{nq^2}{m\xi_0}\boldsymbol{A}(\boldsymbol{r})\int_0^{+\infty} dR\,\exp\left(-\frac{R}{\xi_0}\right) \to -\frac{nq^2}{m}\boldsymbol{A}(\boldsymbol{r})$$

"Soft" vs "hard" superconductors



In 1935, Lev Vasilievich Shubnikov at the Kharkov Institute of Science and Technology in Ukraine discovered that some so called "hard" or type II superconductors (as opposed to "soft" or type I superconductors) exhibit **two critical fields**.



Superconducting magnetization curves of annealed polycrystalline lead and lead-indium alloys at 4.2 K. (A) lead; (B) lead-2.08 wt. % indium; (C) lead-8.23 wt. % indium; (D) lead-20.4 wt.% indium. From Kittel, Introduction to Solid State Physics.

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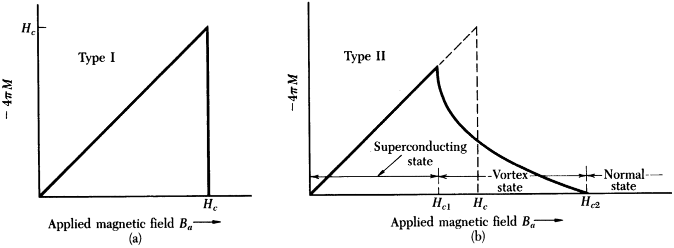


Kittel, Introduction to Solid State Physics

The Meissner effect is incomplete between H_{c1} and H_{c2} ($B \neq 0$).

- H_{c2} is generally much higher than H_c in "soft" superconductors, T_c is also higher. "Hard" superconductors are thus used to generate strong magnetic fields.
- H_{c2} is limited by spin paramagnetism of conduction electrons, see *Clogston, PRL 9, 266 (1962).*
- Except for vanadium, technetium and niobium, "hard" superconductors consist of metallic compounds and alloys.

Discovery of superfluidity

During the 1930s, it was found by several groups that below $T_\lambda = 2.17$ K, helium does not behave like an ordinary liquid.

"*by analogy with superconductors, the helium below the λ-point enters a special state which might be called **superfluid**.*"
Kapitza, Nature 141, 74 (1938).

Kapitza received the Nobel Prize in 1978.

"*the observed type of flow most certainly cannot be treated as laminar or even as ordinary turbulent flow.*"
Allen and Misener, Nature 141, 75 (1938).

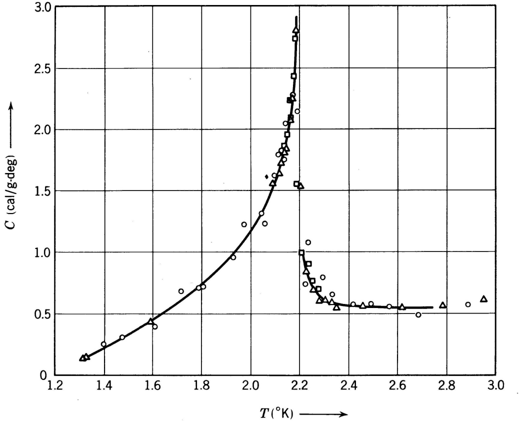
About the history of superfluidity:

Balibar in "History of Artificial Cold, Scientific, Technological and Cultural Issues", Boston Studies in the Philosophy and History of Science 299 (Springer, 2014), pp.93-114

Balibar, J. Low Temp. Phys. 146, 441 (2007).

Lambda point

The specific heat of helium exhibits a sudden change at $T_\lambda = 2.17$ K:

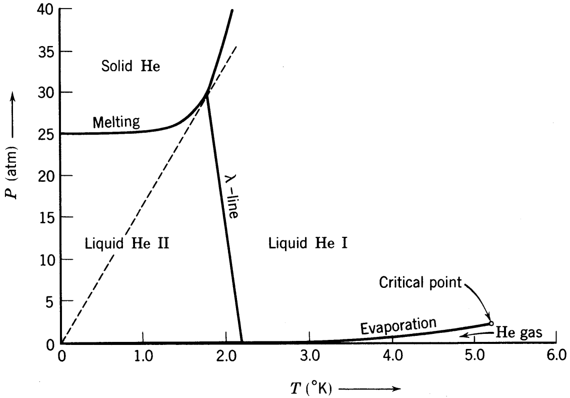


Keesom and Clusius, Proc. Roy. Acad. Amsterdam 35, 307 (1932).

Singularities in the specific heat are generally associated with order-disorder transitions (e.g. ferromagnetic transition).

Phase diagram of helium

Unlike usual liquids, helium does not solidify at low temperatures under the atmospheric pressure:



The horizontal melting curve indicates that the entropy of the liquid is the same as that of the solid, since from the Clausius-Clapeyron equation $\dfrac{dP}{dT} = \dfrac{S_{\rm liq} - S_{\rm sol}}{V_{\rm liq} - V_{\rm sol}} \approx 0.$

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- He II does not follow Fourier's law for the heat current $\boldsymbol{\mathcal{J}} = -\lambda \boldsymbol{\nabla} T$, except in extremely fine slits or capillaries. Actually, the ratio $\mathcal{J}/|\nabla T|$ diverges as $|\nabla T| \to 0$!

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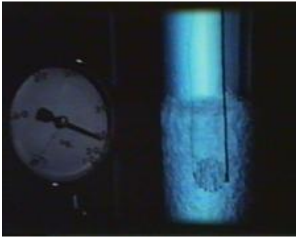
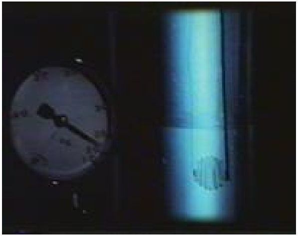
- He II does not boil:



$T > T_\lambda$

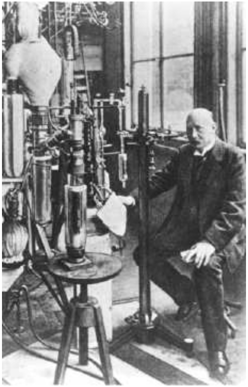


$T < T_\lambda$

"super heat conductivity", Keesom.

Heat in He II is not transported according to classical laws.

K. Onnes observations about liquid helium



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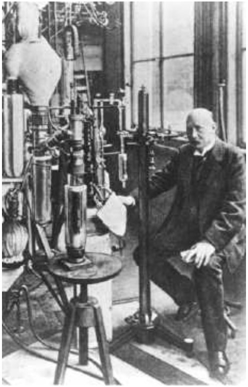

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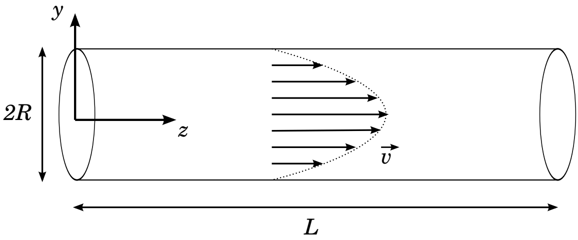
Onnes noted about liquid helium:
"*Just before the lowest temperature [about 1.8 K] was reached, the boiling suddenly stops...*"

About the history of superconductivity:
van Delft&,Kes, Phys. Today, 63, 9, 38 (2010)

`http://www.lps.ens.fr/~balibar/Allen-boiling.mpg`

Intermission: viscous flow



In ordinary liquids, the mass flow Q of liquid through a pipe of length $L \gg R$ is given by **Hagen-Poiseuille law**

$$Q = \rho \frac{\pi R^4 |\Delta P|}{8 \eta L}$$

where ρ is the mass density, $|\Delta P|$ is the pressure drop and η the shear viscosity.

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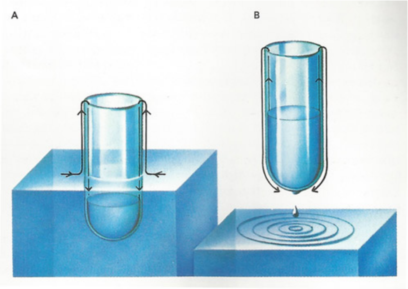
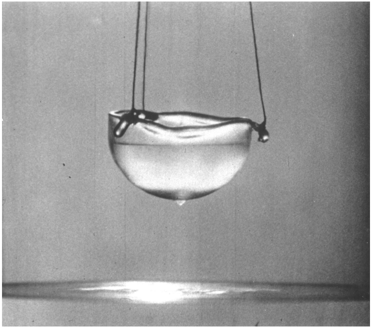
He II does not follow the classical laws of hydrodynamics.

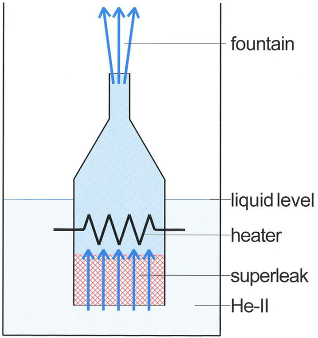
Rollin film

He II flows up over the sides of a beaker and drip off the bottom (for ordinary liquids, the so called Rollin film is clamped by viscosity).





Fountain effect



Allen and Jones, Nature 141, 243 (1938)

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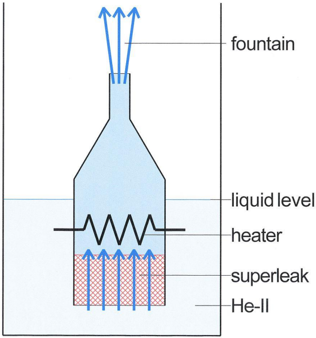

Allen and Jones, Nature 141, 243 (1938)

http://www.lps.ens.fr/~balibar/Allen-fountain.mpg

The superfluid flows from the cooler to the hotter region. From the second law of thermodynamics, we thus conclude that the superfluid carries no heat (no entropy).

Hess-Fairbank effect

Fritz London predicted in 1954 the analog of the Meissner effect for superfluid helium, which was experimentally observed by Hess and Fairbank in 1967.
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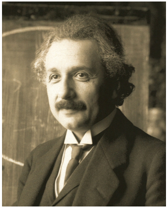
This phenomenon shows that a superfluid is not just a perfect fluid but corresponds to a new thermodynamic state of matter.

Superfluidity and Bose-Einstein condensation



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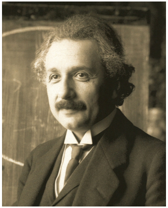

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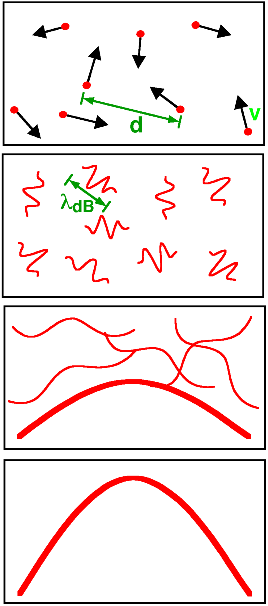
The association between **Bose-Einstein condensation** and superfluidity was first advanced by Fritz London. It was a key idea for developing the microscopic theory of superfluidity and superconductivity.

London, Nature 141, 643 (1938)

What is Bose-Einstein condensation (BEC)?



High Temperature T:

thermal velocity v

density d^{-3}

"Billiard balls"

Low Temperature T:

De Broglie wavelength

$\lambda_{dB}=h/mv \propto T^{-1/2}$

"Wave packets"

$T=T_{crit}$:

Bose-Einstein Condensation

$\lambda_{dB} \approx d$

"Matter wave overlap"

T=0:

Pure Bose condensate

"Giant matter wave"

Illustration of BEC from MIT group

Ideal Bose gas

Let us consider an ideal Bose gas of N noninteracting particles. At $T = 0$, all particles lie in the lowest single-particle energy state $\varepsilon = 0$. The occupancy of this state still remains macroscopic at temperature

$$T < T_c = \frac{2\pi\hbar^2}{m\zeta(3/2)^{2/3}} n^{2/3} \approx 3 \text{ K for helium}$$

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At $T = T_c$, $N_0 = 0$, while at $T = 0$, $N_0 = N$. In liquid helium, $N_0/N \approx 6 - 8\%$ at $T = 0$ due to interactions between atoms.

Two-fluid model

Following the suggestion of Fritz London that superfluidity is related to Bose-Einstein condensation, Laszlo Tisza postulated that **He II contains two disctinct components:**

- a superfluid that carries no entropy (condensate)
- a normal viscous fluid.

This model explained all the observed phenomena and predicted thermomechanical effects like "**temperature waves**". *Tisza, Nature 141, 913 (1938).*

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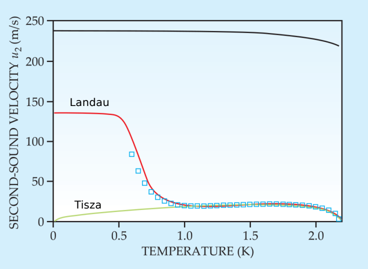
Although Landau did not believe that superfluidity is related to BEC (he never cited F. London!), he developed the two-fluid model based on **"quasiparticle" excitations in quantum fluids**. *Landau, Phys. Rev. 60, 356 (1941).*

This two fluid model was later extended to superconductors. *Gorter, Prog. Low Temp. Phys. 1, 1 (1955)*

Landau vs Tisza and London

Although the two-fluid models of Tisza and Landau were very similar, they led to **different predictions** for the speed u_2 of temperature waves (which Landau called “second sound”) at low temperatures.



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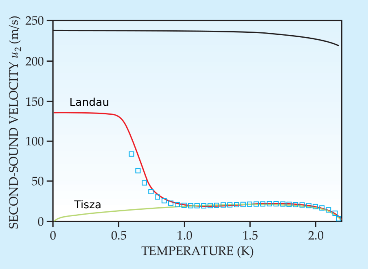

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Tisza considered that the normal fluid was made of non-condensed atoms while for Landau it was made of "quasiparticles". The density of non-condensed atoms is a property of the liquid at rest (ground state) while the density of "quasiparticles" is a property of the superflow (excited state).

Landau's theory of He II

Although helium atoms are strongly interacting, Landau assumed that at low temperatures He II can be described in terms of weakly-interacting **"quasiparticles"**, which do not correspond to material particles but to complex many-body motions (excitations).

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The flow is resistanceless if $V < V_c = \min\left\{\frac{E(p)}{p}\right\}$.

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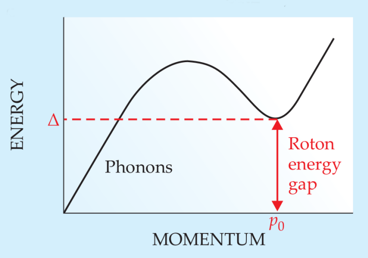
For He II, Landau assumed two different kinds of "quasiparticles":

- **phonons** at low p
  $E(p) \approx c_s p$ (sound waves)
- **rotons** at high p
  $E(p) \approx \Delta + \dfrac{(p - p_0)^2}{2m_0}$



The critical velocity is given by $V_c = \dfrac{\Delta}{p_0} \approx 60$ m s^{-1}. This value has been confirmed by ion propagation experiments.
Ellis and McClintock, Philos. Trans. R. Soc. London, Ser. A, 315, 259 (1985).

Phonons and rotons

In 1947, Bogoliubov calculated the energy of quasiparticles in a **weakly interacting dilute Bose gas** using many-body techniques:

$$E(p) = \sqrt{\left(\frac{p^2}{2m}\right)^2 + p^2 c_s^2} \approx c_s p \text{ at low } p$$

J. Phys.(USSR) 11, 23 (1947)

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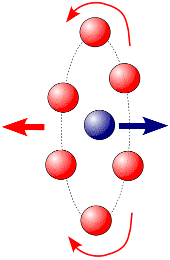
J. Phys.(USSR) 11, 23 (1947)

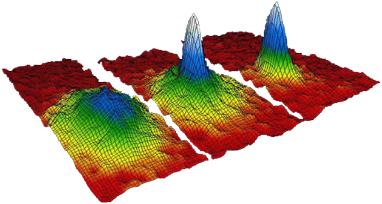
This shows that a BEC of *interacting* particles is superfluid.

Landau thought that rotons are related to vortices. Feynman argued that rotons are **atomic size "smoke rings"**.

Rotons have also been interpreted as a characteristic feature of density fluctuations marking the **onset of crystallization** ("ghosts of Bragg spots", Nozières).
J. Low Temp. Phys. 137, 45 (2004).

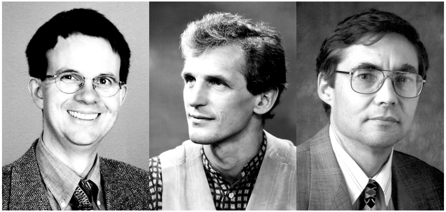


Bose-Einstein condensation in dilute atomic gases



On June 5, 1995, the first dilute BEC was produced by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST-JILA, with $\sim$ 2000 rubidium ^{87}Ru atoms cooled to 170 nK.

Shortly thereafter, Wolfgang Ketterle's team at MIT obtained a BEC of $\sim 5 \times 10^5$ sodium ^{23}Na atoms cooled to 2 μK.



For their achievements, Cornell, Ketterle and Wieman were awarded the 2001 Nobel Prize in Physics.

Since that time, dilute BEC have been produced by other groups using various kinds of atoms.

Flow quantization and vortices

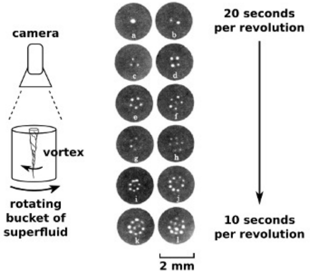
A superfluid is a quantum liquid: its flow is quantized according to the **Onsager-Feynman quantization** condition

$$\oint \boldsymbol{v_s} \cdot d\ell = \frac{Nh}{m} \text{ with } N = 0, 1, \text{ etc.}$$

where $\boldsymbol{v_s}$ is the "superfluid velocity".

A superfluid rotating at angular frequency ω in a bucket of radius R is threaded by $N = \dfrac{2m\pi R^2\omega}{h}$ **quantized vortex lines**, each of which carries an angular momentum $\hbar$.

In between vortices, the flow is "irrotational" $\nabla \times \boldsymbol{v_s} = 0$.



Yarmchuk, Gordon, and Packard, PRL43, 214 (1979).

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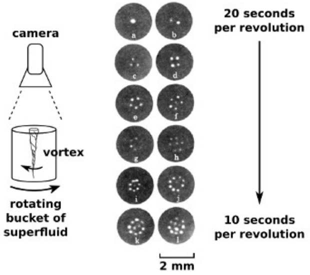
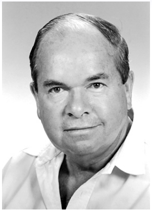


Yarmchuk, Gordon, and Packard, PRL43, 214 (1979).

$\boldsymbol{v_s}$ is not a true velocity: the Onsager-Feynman condition is nothing but the **Bohr-Sommerfeld quantization** $\oint \boldsymbol{p} \cdot d\ell = Nh$ with $\boldsymbol{p} = m\boldsymbol{v_s}$.

Carter and Khalatnikov,Phys.Rev.D45,4536(1992)

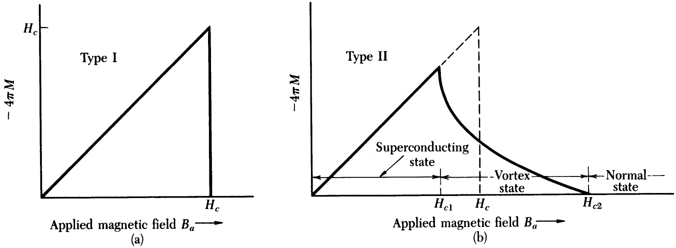
Abrikosov state



In 1957, Alekseï Alekseïevitch Abrikosov predicted that a "hard" superconductor is threaded by a **regular array of magnetic flux tubes** for $H_{c1} < H < H_{c2}$. He was awarded the Nobel Prize in Physics in 2003.

Abrikosov, Soviet Physics JETP 5, 1174 (1957)

Abrikosov state

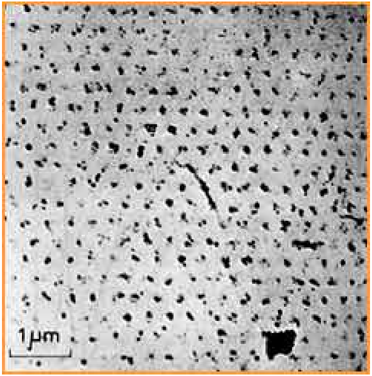


Kittel, Introduction to Solid State Physics

- Below H_{c1}, the magnetic flux is expelled inside the superconductor.
- At $H = H_{c1}$, the first magnetic flux tubes penetrate the superconductor.
- For $H_{c1} < H < H_{c2}$, flux tubes arrange themselves on a regular array with the lattice spacing determined by H (Shubnikov state) .
- At $H = H_{c2}$, the core of magnetic flux tubes overlap and superconductivity disappears.

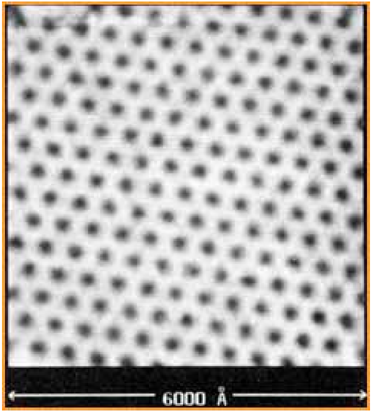
Abrikosov vortex state

Pb-4at%In



Essmann & Trauble, Physics Letters 24A, 526 (1967)

$NbSe_2$



Hess et al., Phys. Rev. Lett. 62, 214 (1989)

Magnetic flux quantization

F. London predicted in 1948 that the magnetic flux inside a superconducting loop must be quantized.

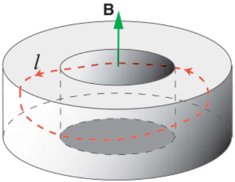


Figure 1.3: Magnetic flux through the hole in a superconductor is quantized.

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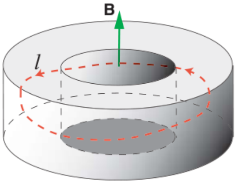

Figure 1.3: Magnetic flux through the hole in a superconductor is quantized.

This was experimentally confirmed in 1961 by Bascom Deaver (PhD) under the supervision of William Fairbank at Stanford University, and independently by Robert Doll and Martin Näbauer at the Low Temperature institute in Hersching (Bavaria).

Deaver & Fairbank, PRL 7, 43 (1961)

Doll & Näbauer, PRL 7, 51 (1961)

See also *100 Years of Superconductivity, published by Horst Rogalla, Peter H. Kes, CRC Press, Taylor & Francis group, 2012, p.161*

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Since $\boldsymbol{j} = nq\boldsymbol{v}$, and
$\oint \left(m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}\right) \cdot \boldsymbol{d\ell} = \frac{m}{nq} \int \nabla \times \boldsymbol{j} \cdot \boldsymbol{dS} + \frac{q}{c} \int \boldsymbol{B} \cdot \boldsymbol{dS}$, we finally obtain

$$\Phi = \int \boldsymbol{B} \cdot \boldsymbol{dS} = N\Phi_0 \text{ with } \Phi_0 = hc/q.$$

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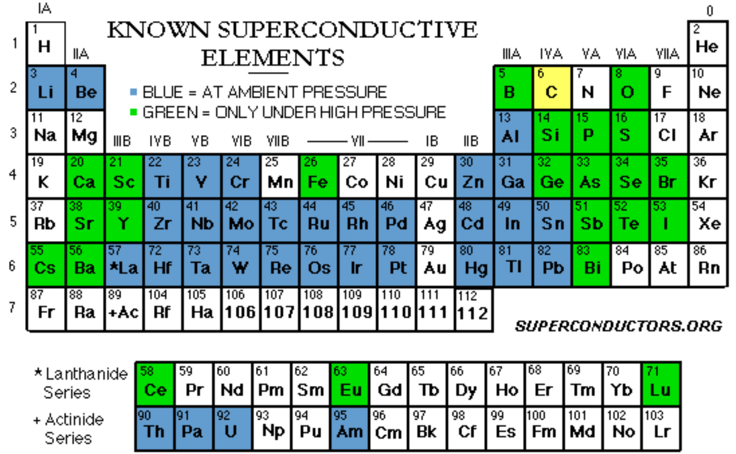
Remarks:

- Φ_0 is called a "fluxoid" or "fluxon".
- $\Phi = \Phi_{\text{ext}} + \Phi_{\text{s}}$. Since Φ_{ext} is not quantized, Φ_{s} must adjust itself accordingly !
- Experimentally $\Phi_0 = hc/(2e)$ therefore the **superconducting particles carry a charge** $q = 2e$.
- The superconducting current will persist unless the flux change.

Superconducting elements

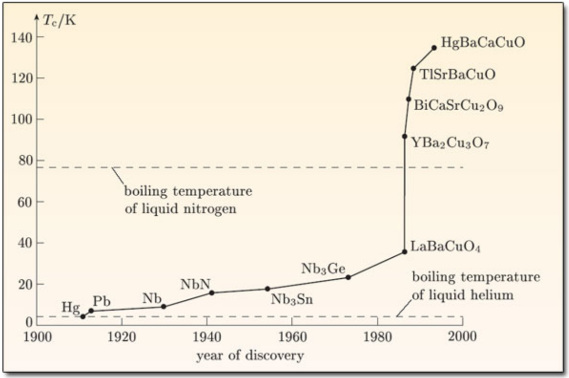


The highest T_c for simple metals is achieved for niobium (Tc=9.2K) and lead (7.2K).

Surprisingly, copper, silver and gold, three of the best metallic conductors, are not superconducting !

Timeline of superconductor discoveries

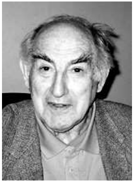
High-T_c cuprate superconductors were discovered by IBM researchers Georg Bednorz and K. Alex Müller in 1986. They were awarded the Nobel Prize in Physics in 1987.



Very recently sulfur hybride has been found to be superconducting (under high pressures) at $T_c = 203$ K.

Drozdov et al., Nature 525, 73 (2015)

Towards a microscopic theory of superconductivity



In 1950, Landau and Ginzburg developed a **phenomenological** theory of superconductivity. Ginzburg shared the 2003 Nobel Prize in Physics with Abrikosov.

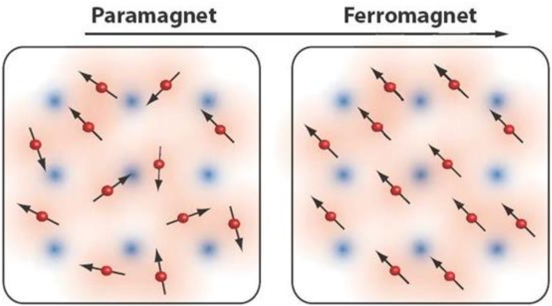
A **microscopic** theory of superconductivity was proposed in 1957 by Bardeen, Cooper and Schrieffer. Gorkov later shown that the Ginzburg-Landau model can be derived from the BCS theory. BCS shared the 1972 Nobel Prize in Physics.

Landau theory of second order phase transitions

Second-order phase transitions are associated with **spontaneous symmetry breaking**, and can be characterized by an **order parameter** η, such that $\eta(T \geq T_c) = 0$ and $\eta(T < T_c) \neq 0$.

Examples:

- liquid-gas phase transition at the critical point $\eta = v_{\mathrm{liq}} - v_{\mathrm{gas}}$
- ferromagnetic-paramagnetic transition $\eta = M$



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$$F(\eta, T) = F(\eta = 0, T) + \alpha\eta^2 + \frac{\beta}{2}\eta^4 + \ldots$$

The equilibrium state of the system at any given T is determined by the minimum of F with respect to η.

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- Above T_c, the only minimum is $\eta = 0$ therefore $\alpha(T) = \alpha_0(T - T_c) + \dots$ with $\alpha_0 > 0$

Landau theory of second order phase transitions

The equilibrium state is such that $\left.\frac{\partial F}{\partial \eta}\right|_T = 0 \Rightarrow 2\alpha\eta + 2\beta\eta^3 = 0$.

Note that this condition is necessary but not sufficient: *F* must be a minimum for the state to be stable.

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- $\eta = 0$ for $T > T_c$,
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Landau's theory thus predicts some kind of **universality** of second-order phase transitions. For instance, $\eta \propto (T_c - T)^{1/2}$.

Note that Landau theory is a **mean field theory**, and does not include spatial fluctuations of the order parameter.

Ginzburg-Landau theory of superconductivity

Ginzburg and Landau extended this theory to superconductivity by postulating the following:

- the order parameter has the nature of a **microscopic wave function** Ψ,
- $|\Psi|^2 = n_s$, where n_s is the density of superconducting particles,
- F_S is invariant under gauge transformations $\Psi \to \Psi \exp(\mathrm{i}\theta)$,
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For T close to T_c, the free energy can be expanded in a Taylor series

$$F_S = F_N + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \dots$$

- $T > T_c$, $\Psi = 0$.
- $T \leq T_c$, $|\Psi|^2 = n_s = -\dfrac{\alpha}{\beta}$ and $F_N - F_S = \dfrac{\alpha^2}{2\beta}$. On the other hand,

  $F_N - F_S = \dfrac{H_c^2}{8\pi}$ therefore $H_c = \sqrt{\dfrac{4\pi\alpha^2}{\beta}} \propto (T_c - T)$.

Ginzburg-Landau theory of superconductivity

Let us consider a more general situation, whereby Ψ is not spatially uniform (fluctuations) and the superconductor is placed in a magnetic field.

For T close to T_c, the free energy becomes

$$F_S = F_N + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m}\left|\frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla}\Psi - q\boldsymbol{A}\Psi\right|^2 + \frac{B^2}{8\pi} + \ldots$$

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Minimizing with respect to Ψ and $\boldsymbol{A}$ yields

- $\frac{1}{2m}\left[\frac{\hbar}{\mathrm{i}}\nabla - q\boldsymbol{A}\right]^2\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0$
- $\boldsymbol{j} = \frac{q}{2m}\frac{\hbar}{\mathrm{i}}\left[\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right] - \frac{q^2}{m}|\Psi|^2\boldsymbol{A}$

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- $-\frac{\hbar^2}{2m}\nabla^2\Psi + \alpha\Psi + \beta\Psi^3 = 0$, or equivalently setting $\varphi \equiv \sqrt{\frac{\beta}{|\alpha|}}\Psi$

$$\xi^2\nabla^2\varphi + \varphi(1 - \varphi^2) = 0 \text{ with } \xi \equiv \sqrt{\frac{\hbar^2}{2m|\alpha|}}$$

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For a semi-infinite superconductor in $x \geq 0$, $\varphi(x) = \tanh\left(\frac{x}{\sqrt{2}\xi}\right)$.

Therefore $\varphi(0) = 0$ at the boundary between the normal and superconducting phases, while deep inside the superconductor $\varphi(x \to +\infty) = 1$ so that $\Psi(x \to +\infty) = \sqrt{|\alpha|/\beta}$.

The **coherence length** ξ is the characteristic distance over which $\Psi(\boldsymbol{r})$ fluctuates.

London penetration length

Let us now assume $\Psi(\boldsymbol{r}) = \sqrt{n_s} = \sqrt{|\alpha|/\beta}$ (no spatial fluctuations) in a weak magnetic field $B \ll H_c$.

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The second Ginzburg-Landau's equation reduces to London's equation :

$$\boldsymbol{j} = \frac{q}{2m}\frac{\hbar}{\mathrm{i}}\left[\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right] - \frac{q^2}{m}|\Psi|^2\boldsymbol{A} = -\frac{n_s q^2}{m}\boldsymbol{A}$$

$$\Rightarrow \lambda_L \nabla^2 \boldsymbol{B} = \boldsymbol{B} \text{ with } \lambda_L = \sqrt{\frac{mc^2}{4\pi n_s q^2}} = \sqrt{\frac{mc^2\beta}{4\pi|\alpha|q^2}}$$

The **London penetration length** λ_L is the characteristic distance over which $\boldsymbol{B}$ penetrates the superconductor.

Two characteristic length scales

The Ginzburg-Landau theory predicts that both λ_L and ξ scale like $|\alpha|^{-1/2} \propto (T_c - T)^{-1/2}$ but their ratio is constant

$$\kappa \equiv \frac{\lambda_L}{\xi} = \frac{mc}{q\hbar}\sqrt{\frac{\beta}{2\pi}}$$

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$$\kappa \equiv \frac{\lambda_L}{\xi} = \frac{mc}{q\hbar}\sqrt{\frac{\beta}{2\pi}}$$

One can show that

- if $\kappa < 1/\sqrt{2}$, the superconductor is "soft",
- if $\kappa > 1/\sqrt{2}$, the superconductor is "hard".

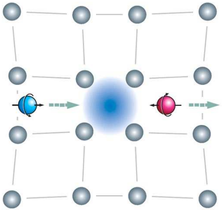
Roughly speaking, at H_{c1} the first fluxoid nucleates. It carries a quantum flux Φ_0: the magnetic field inside is $\sim H_{c1}$ and extends over a distance $\sim \lambda_L$. At H_{c2}, fluxoids are the most densely packed with a spacing $\sim \xi$ and the magnetic field penetrates almost uniformly the superconductor.

Therefore $H_{c1} \sim \dfrac{\Phi_0}{\pi\lambda_L^2}$ and $H_{c2} \sim \dfrac{\Phi_0}{\pi\xi^2}$. Note that $\dfrac{H_{c2}}{H_{c1}} \sim \dfrac{\lambda_L}{\xi}$.

If $\xi \gtrsim \lambda_L$, fluxoids cannot form.

BCS theory of superconductivity

The discovery of the **isotope effect**, $T_c \propto M^{-\alpha}$, suggested that crystal lattice dynamics play a role in superconductivity.



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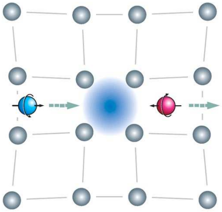

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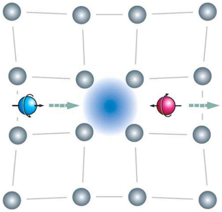

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The pairing scenario suggested that fermionic atoms could also be superfluid. Osheroff found in 1971 that ^{3}He is superfluid below 2.5 mK. The first dilute fermionic superfluids were produced in 2003.

Effective electron-electron interaction

Two electrons in vacuum repel each other due to the instantaneous Coulomb interaction $V(\boldsymbol{r_1}, t_1, \boldsymbol{r_2}, t_2) = \dfrac{e^2}{r}\delta(t)$ with $r = |\boldsymbol{r_1} - \boldsymbol{r_2}|$ and $t = t_1 - t_2$.

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$$\widetilde{V}(\boldsymbol{q}, \omega) = \frac{1}{\Omega} \int dt \int d^3r \, V(\boldsymbol{r}) e^{-\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r} + \omega t)} = \frac{4\pi e^2}{\Omega q^2}.$$

Two conduction electrons in a solid interact with other electrons and with ions. Their "bare" interaction is thus "dressed" by the medium.

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Typical scales in a solid:

- conduction electrons of density n (Fermi gas)
  Fermi energy $\varepsilon_F = \frac{1}{2}mv_F^2$ where $v_F = \frac{\hbar k_F}{m}$ is the Fermi velocity and $k_F = (3\pi^2 n)^{1/3}$
- low-energy longitudinal lattice vibrations (phonons)

  ion plasma frequency $\omega_p = \sqrt{\frac{4\pi Z^2 e^2 n_{\mathrm{I}}}{M}}$

Bardeen-Pines interaction

Approximating a solid by a "jelium", the **effective interaction** between electrons is approximately given by ($q \ll k_F$, $\omega \ll c_s k_F$, $c_s \ll v_F$)

$$V_{\mathrm{eff}}(\boldsymbol{q}, \omega) = \underbrace{\frac{4\pi e^2}{q^2 + q_{\mathrm{TF}}^2}}_{\text{screening}} + \underbrace{\frac{4\pi e^2}{q^2 + q_{\mathrm{TF}}^2} \frac{\omega(\boldsymbol{q})^2}{\omega^2 - \omega(\boldsymbol{q})^2}}_{\text{polarization}}$$

where $\omega(\boldsymbol{q}) \approx c_s q$, $c_s = \omega_p / q_{\mathrm{TF}}$ is the sound speed, and $q_{\mathrm{TF}} = \sqrt{4\pi e^2 \frac{\partial n}{\partial \mu}}$ is the Thomas-Fermi wave vector.

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- **Charge screening** makes the Coulomb interaction much less repulsive at large distances $V(r) = \frac{e^2}{r} \to V_{\text{eff}}(r) = \frac{e^2}{r} e^{-q_{\text{TF}} r}$
- **Polarization of the ion lattice** leads to retarded interaction: the distortion of the lattice induced by the first electron is felt at a later time by the second electron.

The effective electron-electron interaction induced by the polarization of the ion lattice is **attractive** for $\omega < \omega(\boldsymbol{q})$ and repulsive otherwise.

Cooper problem (1956)

Let us consider two conduction electrons of opposite spins on the Fermi surface interacting through an instantaneous effective attractive pairing interaction of the form

$$V_{\text{eff}}(\boldsymbol{r_1}, t_1, \boldsymbol{r_2}, t_2) = -V_0 \delta(\boldsymbol{r}) \delta(t) \text{ with } V_0 > 0.$$

where $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$ and $t = t_1 - t_2$.

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where $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$ and $t = t_1 - t_2$.

Its Fourier transform can be easily calculated

$$\widetilde{V}_{\text{eff}}(\boldsymbol{q}, \omega) = \frac{1}{\Omega} \int dt \int d^3 r \, V_{\text{eff}}(\boldsymbol{r}, t) e^{-\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{r} + \omega t)} = \frac{-V_0}{\Omega}.$$

This interaction leads to arbitrarily large energy transfers and will thus need to be regularized.

In the following, we shall consider a time-independent interaction of the more familiar kind $V_{\text{eff}}(\boldsymbol{r_1}, \boldsymbol{r_2}) = V_{\text{eff}}(\boldsymbol{r})$ with a suitable prescription to eliminate divergences.

Cooper problem (1956)

The two-electron wave function can be expanded into plane waves

$$\psi(\boldsymbol{r_1}\sigma_1, \boldsymbol{r_2}\sigma_2) = \sum_{\boldsymbol{k_1},\boldsymbol{k_2}} \widetilde{\psi}(\boldsymbol{k_1}, \boldsymbol{k_2}) e^{\mathrm{i}(\boldsymbol{k_1}\cdot\boldsymbol{r_1}+\boldsymbol{k_2}\cdot\boldsymbol{r_2})} \chi(\sigma_1, \sigma_2)$$

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Let us introduce the center-of-mass coordinate $\boldsymbol{R} = (\boldsymbol{r_1} + \boldsymbol{r_2})/2$ and relative coordinate $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$, total wave vector $\boldsymbol{K} = \boldsymbol{k_1} + \boldsymbol{k_2}$ and relative wave vector $\boldsymbol{k} = \boldsymbol{k_1} - \boldsymbol{k_2}$, the wave function can be equivalently

$$\psi(\boldsymbol{R}, \boldsymbol{r}) = \sum_{\boldsymbol{k_1},\boldsymbol{k_2}} \widetilde{\psi}(\boldsymbol{K}, \boldsymbol{k}) e^{\mathrm{i}(\boldsymbol{K}\cdot\boldsymbol{R}+\boldsymbol{k}\cdot\boldsymbol{r})}\chi(\sigma_1, \sigma_2)$$

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Assumptions:

- The electron pair has zero net momentum $\boldsymbol{K} = 0$

  Note that $\boldsymbol{k}\cdot\boldsymbol{r} = \boldsymbol{k}\cdot\boldsymbol{r_1} - \boldsymbol{k}\cdot\boldsymbol{r_2}$: electrons have opposite momenta.
- Electrons have opposite spins $\chi = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$

Cooper problem (1956)

In the following, we shall focus on the spatial part of the wave function, which thus becomes $\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \widetilde{\psi}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, and $\widetilde{\psi}(-\boldsymbol{k}) = \widetilde{\psi}(\boldsymbol{k})$ as a consequence of the Pauli principle.

Thermodynamic limit:

Let us take the electron number $N \to +\infty$, the volume $\Omega \to +\infty$ but N/Ω finite. Since bulk properties do not depend on the shape of the system, let us consider a cubic box of size L ($\Omega = L^3$).

Born- von Karman periodic boundary conditions:

$$\psi(x+L, y+L, z+L) = \psi(x, y, z)$$

$$\Rightarrow k_x = \frac{2\pi N_x}{L},\ k_y = \frac{2\pi N_y}{L}, k_z = \frac{2\pi N_z}{L}$$

In the thermodynamic limit, $\displaystyle \frac{1}{\Omega}\sum_{\boldsymbol{k}} \to \int \frac{d^3k}{(2\pi)^3}$

Cooper problem (1956)

The two-electron Schrödinger equation can be written as

$$\left[-\frac{\hbar^2}{2m}\left(\nabla_1+\nabla_1\right)+V_{\mathrm{eff}}(\boldsymbol{r}_1,\boldsymbol{r}_2)\right]\psi(\boldsymbol{r}_1,\boldsymbol{r}_2)=\left(\epsilon+2\frac{\hbar^2k_F^2}{2m}\right)\psi(\boldsymbol{r}_1,\boldsymbol{r}_2)$$

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Expanding the two-electron wave function into plane waves leads to

$$\int\frac{d^3k'}{(2\pi)^3}\left[\frac{\hbar^2k'^2}{m}+V_{\text{eff}}(\boldsymbol{r})\right]\widetilde{\psi}(\boldsymbol{k'})e^{\mathrm{i}\boldsymbol{k'}\cdot\boldsymbol{r}}=\left(\epsilon+\frac{\hbar^2k_F^2}{m}\right)\int\frac{d^3k'}{(2\pi)^3}\widetilde{\psi}(\boldsymbol{k'})e^{\mathrm{i}\boldsymbol{k'}\cdot\boldsymbol{r}}$$

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Multiplying by $e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ and integrating over $\boldsymbol{r}$ yields

$$\frac{\hbar^2 k^2}{m}\widetilde{\psi}(\boldsymbol{k}) + \Omega\int \frac{d^3k'}{(2\pi)^3}\widetilde{V}_{\text{eff}}(\boldsymbol{k'} - \boldsymbol{k})\widetilde{\psi}(\boldsymbol{k'}) = (\epsilon + 2\varepsilon_F)\,\widetilde{\psi}(\boldsymbol{k})$$

where $\widetilde{V}_{\text{eff}}(\boldsymbol{q}) = \dfrac{1}{\Omega}\displaystyle\int d^3r\, V_{\text{eff}}(\boldsymbol{r})e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$ and $\varepsilon_F = \dfrac{\hbar^2 k_F^2}{2m}$.

Note: $\displaystyle\int d^3r\, e^{\mathrm{i}(\boldsymbol{k'} - \boldsymbol{k})\cdot\boldsymbol{r}} = (2\pi)^3\delta(\boldsymbol{k'} - \boldsymbol{k})$

Cooper problem (1956)

How to regularize the effective interaction?

- We have seen that the effective interaction is attractive only for electrons with energies $\omega < \omega(\boldsymbol{q})$.
- Moreover, electrons must have an energy greater than ε_F.

We shall therefore regularize the effective pairing interaction as follows:

$$\widetilde{V}_{\mathrm{eff}}(\boldsymbol{k'}-\boldsymbol{k}) = \begin{cases} -\dfrac{V_0}{\Omega} & \text{if } \left|\dfrac{\hbar^2 k^2}{2m} - \varepsilon_F\right| < \hbar\omega_D \text{ and } \left|\dfrac{\hbar^2 k'^2}{2m} - \varepsilon_F\right| < \hbar\omega_D \,, \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_D = q_D c_s$ is the Debye frequency and $q_D = (6\pi^2 n_{\mathrm{I}})^{1/3}$.

Cooper problem (1956)

The Schrödinger equation

$$\left[\frac{\hbar^2 k^2}{m} - \epsilon - 2\varepsilon_F\right]\widetilde{\psi}(\boldsymbol{k}) = -\Omega \int \frac{d^3k'}{(2\pi)^3}\widetilde{V}(\boldsymbol{k'} - \boldsymbol{k})\widetilde{\psi}(\boldsymbol{k'})$$

thus reduces to

$$\left[\frac{\hbar^2 k^2}{m} - \epsilon - 2\varepsilon_F\right]\widetilde{\psi}(\boldsymbol{k}) = -V_0 \int_D \frac{d^3k'}{(2\pi)^3}\widetilde{\psi}(\boldsymbol{k'}) \equiv -V_0 C$$

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$$\widetilde{\psi}(\boldsymbol{k}) = -\frac{V_0 C}{2\zeta(\boldsymbol{k}) - \epsilon} \text{ with } \zeta(\boldsymbol{k}) \equiv \frac{\hbar^2 k^2}{2m} - \varepsilon_F$$

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Integrating over $\boldsymbol{k}$ leads to the self-consistency condition

$$-\frac{1}{V_0} = \int_D \frac{d^3k}{(2\pi)^3}\frac{1}{2\zeta - \epsilon} = \int_0^{\hbar\omega_D} \frac{\mathcal{N}(\zeta)d\zeta}{2\zeta - \epsilon}$$

with $\mathcal{N}(\zeta) = \int \frac{d^3k}{(2\pi)^3}\delta(\zeta - \zeta(\boldsymbol{k}))$ is the density of states.

Cooper problem (1956)

Considering that $\hbar\omega_D \ll \varepsilon_F$, we find

$$-\frac{1}{V_0} = \int_0^{\hbar\omega_D} \frac{\mathcal{N}(\zeta)d\zeta}{2\zeta - \epsilon} \approx \mathcal{N}(0) \int_0^{\hbar\omega_D} \frac{d\zeta}{2\zeta - \epsilon} = \frac{1}{2}\mathcal{N}(0)\ln\left(\frac{\epsilon}{\epsilon - 2\hbar\omega_D}\right)$$

Assuming $|\epsilon| \ll \hbar\omega_D$, we finally obtain

$$\epsilon \approx -2\hbar\omega_D \exp\left(-\frac{2}{\mathcal{N}(0)\,V_0}\right)$$

- $\epsilon < 0$ therefore the Cooper pair is bound, even for arbitrarily small interaction.
- The bound state would not exist without the Fermi sea ($\mathcal{N}(0) = 0$).
- ϵ cannot be obtained using perturbation theory.

Cooper pair

The wave function of a Cooper pair of electrons is given by

$$\psi(\boldsymbol{r}) = \Omega \int \frac{d^3k}{(2\pi)^3}\, \widetilde{\psi}(\boldsymbol{k}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = -\Omega V_0 C \int \frac{d^3k}{(2\pi)^3}\, \frac{e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{2\xi(\boldsymbol{k}) - \epsilon}.$$

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The mean-square radius is

$$\langle r^2 \rangle = \frac{\int d^3r\, |\psi(\boldsymbol{r})|^2 r^2}{\int d^3r\, |\psi(\boldsymbol{r})|^2} = \frac{4}{3} \frac{\hbar^2 v_F^2}{\epsilon^2}$$

Therefore the typical size of a Cooper pair is

$$\xi_0 = \sqrt{\langle r^2 \rangle} = \frac{2}{\sqrt{3}} \frac{\hbar v_F}{\epsilon} \sim \frac{\varepsilon_F}{\epsilon} \ell \gg \ell$$

where $\ell \sim n^{-1/3}$ is the mean inter electron spacing.

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The Fermi sea is unstable against the formation of pairs. The presence of the other electrons cannot be ignored.

BCS theory

The Cooper model suggests that the ground state of a superconductor should be constructed from electron pairs

$$\Psi_N = \mathcal{A}\Big\{\psi(\boldsymbol{r_1}\sigma_1, \boldsymbol{r_2}\sigma_2)\cdots\psi(\boldsymbol{r_{N-1}}\sigma_{N-1}, \boldsymbol{r_N}\sigma_N)\Big\}$$

where $\mathcal{A}$ is to indicate that the wave function must be properly antisymmetrized as imposed by the Pauli principle. Expanding each pair wave function into plane waves, we obtain

$$\Psi_N = \sum_{\boldsymbol{k_1}\cdots\boldsymbol{k_{N/2}}} \widetilde{\psi}(\boldsymbol{k_1})\cdots\widetilde{\psi}(\boldsymbol{k_{N/2}})\mathcal{A}\Big\{e^{\mathrm{i}\boldsymbol{k_1}\cdot(\boldsymbol{r_1}-\boldsymbol{r_2})}\ldots e^{\mathrm{i}\boldsymbol{k_{N/2}}\cdot(\boldsymbol{r_{N-1}}-\boldsymbol{r_N})}\Big\}\chi(\sigma_1\cdots\sigma_N)$$

Algebraic manipulations using this representation are quite cumbersome. It is much simpler to use the second quantization notation.

BCS theory

$$
\begin{aligned}
|\Psi_N\rangle &= \sum_{\boldsymbol{k_1}\cdots\boldsymbol{k_{N/2}}} \widetilde{\psi}(\boldsymbol{k_1})\cdots\widetilde{\psi}(\boldsymbol{k_{N/2}})\, c^\dagger_{\boldsymbol{k_1}\uparrow} c^\dagger_{-\boldsymbol{k_1}\downarrow}\cdots c^\dagger_{\boldsymbol{k_{N/2}}\uparrow} c^\dagger_{-\boldsymbol{k_{N/2}}\downarrow}|\Psi_0\rangle \\
&= \left[\sum_{\boldsymbol{k}} \widetilde{\psi}(\boldsymbol{k})\, c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\right]^{N/2} |\Psi_0\rangle
\end{aligned}
$$

where $|\Psi_0\rangle$ denotes the vacuum state (no particles), while $c^\dagger_{\boldsymbol{k}\sigma}$ ($c_{\boldsymbol{k}\sigma}$) is the Fermi creation (annihilation) operator of a particle with wave vector $\boldsymbol{k}$ and spin σ.

The antisymmetry of the wave function is guaranteed by the anticommutation rules:

$$
\{c^\dagger_{\boldsymbol{k}\sigma}, c_{\boldsymbol{k'}\sigma'}\} \equiv c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k'}\sigma'} + c_{\boldsymbol{k'}\sigma'} c^\dagger_{\boldsymbol{k}\sigma} = \delta_{\boldsymbol{kk'}}\delta_{\sigma\sigma'}
$$

$$
\{c^\dagger_{\boldsymbol{k}\sigma}, c^\dagger_{\boldsymbol{k'}\sigma'}\} = 0\,, \quad \{c_{\boldsymbol{k}\sigma}, c_{\boldsymbol{k'}\sigma'}\} = 0.
$$

$|\Psi_N\rangle$ contains a huge number of terms of order $\sim 10^{10^{22}}$!

BCS theory

Bardeen, Cooper and Schrieffer adopted a **mean-field approach**: the occupancy of the state $\boldsymbol{k}$ only depends on the *average* occupancy of the other states.

$$|\Psi_{BCS}\rangle = \prod_{\boldsymbol{k}} \left[u(\boldsymbol{k}) + v(\boldsymbol{k}) c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \right] |\Psi_0\rangle$$

The normalization of the wave function $\langle \Psi_{BCS} | \Psi_{BCS} \rangle = 1$ yields $|u(\boldsymbol{k})|^2 + |v(\boldsymbol{k})|^2 = 1$.

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The price to be paid is that $|\Psi_{BCS}\rangle$ does not define a state with a well-defined number N of electrons:

$$\overline{N} = \langle\Psi_{BCS}| \sum_{\boldsymbol{k}\sigma} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} |\Psi_{BCS}\rangle = 2 \sum_{\boldsymbol{k}} |v(\boldsymbol{k})|^2 \ .$$

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However, the fluctuations are vanishing small in the thermodynamic limit: $\dfrac{\sqrt{\overline{(\delta N)^2}}}{\overline{N}} \sim \overline{N}^{-1/2} \to 0$ as $N \to +\infty$.

BCS theory

The BCS Hamiltonian is

$$H = \sum_{\boldsymbol{k}\sigma} \frac{\hbar^2 k^2}{2m} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k},\boldsymbol{k'}} \widetilde{V}_{\mathrm{eff}}(\boldsymbol{k'} - \boldsymbol{k}) c^\dagger_{\boldsymbol{k'}\uparrow} c^\dagger_{-\boldsymbol{k'}\downarrow} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}$$

Since the electron number is not conserved, the ground state is found by minimizing $\langle \Psi_{BCS} | H | \Psi_{BCS} \rangle - \mu \langle \Psi_{BCS} | N | \Psi_{BCS} \rangle$, where the Lagrange multiplier μ is the chemical potential.

BCS theory

The BCS Hamiltonian is

$$H = \sum_{\boldsymbol{k}\sigma} \frac{\hbar^2 k^2}{2m} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k},\boldsymbol{k}'} \widetilde{V}_{\text{eff}}(\boldsymbol{k}' - \boldsymbol{k}) c^\dagger_{\boldsymbol{k}'\uparrow} c^\dagger_{-\boldsymbol{k}'\downarrow} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}$$

Since the electron number is not conserved, the ground state is found by minimizing $\langle \Psi_{BCS}|H|\Psi_{BCS}\rangle - \mu\langle \Psi_{BCS}|N|\Psi_{BCS}\rangle$, where the Lagrange multiplier μ is the chemical potential. The solution is

$$u(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\zeta(\boldsymbol{k})}{E(\boldsymbol{k})}}, \quad v(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\zeta(\boldsymbol{k})}{E(\boldsymbol{k})}}$$

$$\zeta(\boldsymbol{k}) = \frac{\hbar^2 k^2}{2m} - \mu\,, \quad E(\boldsymbol{k}) = \sqrt{\zeta(\boldsymbol{k}^2) + \Delta(\boldsymbol{k})^2}$$

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} \widetilde{V}_{\text{eff}}(\boldsymbol{k}' - \boldsymbol{k}) \frac{\Delta(\boldsymbol{k}')}{2E(\boldsymbol{k}')}$$

Remark: μ is determined by the condition $N = 2\sum_{\boldsymbol{k}} |v(\boldsymbol{k})|^2$.

BCS theory

- In the absence of pairing $\widetilde{V}_{\text{eff}}(\boldsymbol{k'} - \boldsymbol{k}) = 0$, we find

$$\Delta(\boldsymbol{k}) = 0\,, \quad E(\boldsymbol{k}) = |\zeta(\boldsymbol{k})|$$

$$u(\boldsymbol{k}) = \begin{cases} 1 & \text{if } k > k_F\,, \\ 0 & \text{otherwise} \end{cases} \qquad v(\boldsymbol{k}) = \begin{cases} 0 & \text{if } k > k_F\,, \\ 1 & \text{otherwise} \end{cases}$$

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- Using the effective pairing interaction assuming $\mu \approx \varepsilon_F$ and $\hbar\omega_D \ll \varepsilon_F$ (weak coupling approximation)

$$\widetilde{V}_{\text{eff}}(\boldsymbol{k'}-\boldsymbol{k}) = \begin{cases} -\dfrac{V_0}{\Omega} & \text{if } \left|\dfrac{\hbar^2 k^2}{2m} - \varepsilon_F\right| < \hbar\omega_D \text{ and } \left|\dfrac{\hbar^2 k'^2}{2m} - \varepsilon_F\right| < \hbar\omega_D\,, \\ \quad 0 & \text{otherwise} \end{cases}$$

$$\boxed{\Delta(\boldsymbol{k}) = \Delta_0 \approx 2\hbar\omega_D \exp\left(-\frac{1}{\mathcal{N}(0)\,V_0}\right)}$$

Condensation energy $F_N - F_S = -\dfrac{1}{2}\mathcal{N}(0)\Delta^2$

BCS theory

At finite temperature, the BCS gap equation becomes

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} \widetilde{V}_{\text{eff}}(\boldsymbol{k}' - \boldsymbol{k}) \frac{\Delta(\boldsymbol{k}')}{2E(\boldsymbol{k}')} \tanh\left[\frac{E(\boldsymbol{k}')}{2k_B T}\right].$$

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Using the effective pairing interaction of Cooper, we find ($\gamma \approx 0.577$ is the Euler-Mascheroni constant)

$$k_B T_c = \frac{2\exp(\gamma)}{\pi} \hbar\omega_D \exp\left(-\frac{1}{\mathcal{N}(0)V_0}\right)$$

The BCS theory predicts the **universal relation** $\dfrac{\Delta_0}{T_c} = \dfrac{\pi}{\exp(\gamma)}$.

Moreover, $\dfrac{\Delta(T)}{\Delta_0}$ depends solely on $\dfrac{T}{T_c}$, and is approximately given by

$$\Delta(T) \approx \Delta_0 \sqrt{1 - \left(\frac{T}{T_c}\right)^{\delta}} \quad \text{with } \delta \approx 3.23.$$

S. Goriely, Nucl. Phys. A 605, 28 (1996).

BCS theory

The BCS theory predicts a discontinuity in the heat capacity at T_c:

$$\frac{C_S - C_N}{C_N} \approx \frac{3}{2}\delta \exp(-2\gamma) \approx 1.5.$$

The behavior of the heat capacity (per unit volume) at *any* temperature T is well reproduced by the following formulas:

$$C(T) \approx \frac{3}{2} n\, R_{00}(u) \left[1 - \exp\left(-\frac{T}{T_{cl}}\right)\right]$$

$$R_{00}(u) = \left[a_0 + \sqrt{(a_1)^2 + (a_2 u)^2}\right]^{\alpha} \exp\left(b_0 - \sqrt{b_1^2 + u^2}\right)$$

where $u = \sqrt{1-\tau}\left(c_0 - \frac{c_1}{\sqrt{\tau}} + \frac{c_2}{\tau}\right)$, $\tau = \frac{T}{T_c}$, and $T_{cl} = \frac{3\varepsilon_F}{k_B\,\pi^2}$.

The values of the parameters can be found in

Pastore,Chamel,Margueron,MNRAS 448, 1887 (2015)

Levenfish&Yakovlev, Astron. Rep., 38, 247 (1994)

superfluid	degenerate	classical
$T \ll T_c$	$T > T_c$ and $T \ll T_F$	$T \gg T_F$
$C \propto \exp(-c_2 T_c/T)$	$C \propto T$	$C \propto (3/2)n$

Theory of inhomogeneous superconductors

So far, the electron gas was assumed to be uniform. Accounting for inhomogeneities (e.g. impurities, defects) using the same effective pairing interaction $V_{\mathrm{eff}}(\boldsymbol{r_1}, \boldsymbol{r_2}) = -V_0\delta(\boldsymbol{r_1} - \boldsymbol{r_2})$ leads to the **Bogoliubov-de Gennes equations**:

$$\begin{pmatrix} h_0(\boldsymbol{r}) + U(\boldsymbol{r}) & \Delta(\boldsymbol{r}) \\ \Delta(\boldsymbol{r}) & -h_0(\boldsymbol{r}) - U(\boldsymbol{r}) \end{pmatrix} \begin{pmatrix} \varphi_{1k}(\boldsymbol{r}) \\ \varphi_{2k}(\boldsymbol{r}) \end{pmatrix} = E_k \begin{pmatrix} \varphi_{1k}(\boldsymbol{r}) \\ \varphi_{2k}(\boldsymbol{r}) \end{pmatrix}$$

$$h_0(\boldsymbol{r}) \equiv -\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}q}{\hbar c}\boldsymbol{A}\right)^2 - \mu \text{ is the kinetic operator}$$

$$U(\boldsymbol{r}) = -V_0 n(\boldsymbol{r}) \text{ is the } \textbf{mean-field potential}$$

$$\Delta(\boldsymbol{r}) = -\frac{1}{2}V_0\tilde{n}(\boldsymbol{r}) \text{ is the } \textbf{pair potential}$$

$$n(\boldsymbol{r}) = \sum_k \left\{ f_k|\varphi_{1k}(\boldsymbol{r})|^2 + (1 - f_k)|\varphi_{2k}(\boldsymbol{r})|^2 \right\} \text{ is the } \textbf{“normal” density}$$

$$\tilde{n}(\boldsymbol{r}) = \sum_k (2f_k - 1)\varphi_{2k}(\boldsymbol{r})\varphi_{1k}(\boldsymbol{r})^* \text{ is the } \textbf{“abnormal” density}$$

$$f_k = \frac{1}{1 + \exp(E_k/k_B T)} \text{ is the Fermi occupation factor}$$

Theory of inhomogeneous superconductors

In the limiting case of a uniform superconductor with $\boldsymbol{A} = 0$, the quasiparticle wavefunction can be written as

$$\varphi_{1k}(\boldsymbol{r}) = \frac{1}{\sqrt{\Omega}} u(\boldsymbol{k}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} , \qquad \varphi_{2k}(\boldsymbol{r}) = \frac{1}{\sqrt{\Omega}} v(\boldsymbol{k}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

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The Bogoliubov-de Gennes equation thus reduces to

$$\begin{pmatrix} \zeta(\boldsymbol{k}) & \Delta \\ \Delta & -\zeta(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} u(\boldsymbol{k}) \\ v(\boldsymbol{k}) \end{pmatrix} = E(\boldsymbol{k}) \begin{pmatrix} u(\boldsymbol{k}) \\ v(\boldsymbol{k}) \end{pmatrix}$$

with $\zeta(\boldsymbol{k}) \equiv \dfrac{\hbar^2 k^2}{2m} - \mu - V_0 n \approx \dfrac{\hbar^2 k^2}{2m} - \varepsilon_F$.

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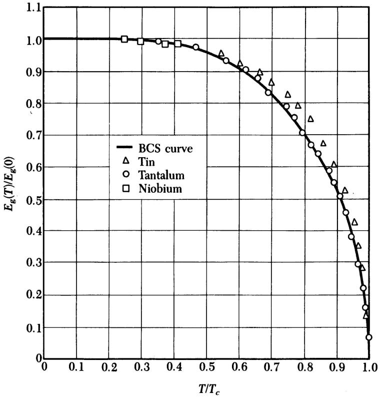
The solutions are $u(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\zeta(\boldsymbol{k})}{E(\boldsymbol{k})}}$, $v(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\zeta(\boldsymbol{k})}{E(\boldsymbol{k})}}$

with $E(\boldsymbol{k}) = \sqrt{\zeta(\boldsymbol{k})^2 + \Delta^2}$, and

$$\Delta = -\frac{V_0}{\Omega} \sum_{\boldsymbol{k}} (2f_k - 1) u(\boldsymbol{k}) v(\boldsymbol{k}) = \frac{V_0}{\Omega} \sum_{\boldsymbol{k}} \frac{\Delta}{2E(\boldsymbol{k})} \tanh\left[\frac{E(\boldsymbol{k})}{2k_B T}\right]$$

Beyond BCS

The BCS theory has been very successful in describing so called conventional superconductors with low T_c (weak coupling).

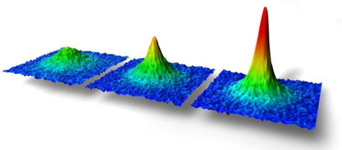


Kittel, Introduction to Solid State Physics

The BCS mean-field approach was later reformulated and extended using quantum field theory methods.

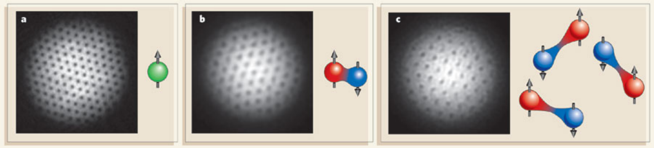
Fermionic condensates: from BEC to BCS

On December 16, 2003, the first dilute fermionic condensate was produced by Deborah Jin at JILA with 500 000 potassium ^{40}K atoms cooled to 50 nK.



By varying the pairing interaction with a magnetic field, it is possible to study the **crossover from a BEC to a BCS state**.

Leggett&Zhang in "The BCS-BEC Crossover and the Unitary Fermi Gas", Lecture Notes in Physics 836 (Springer, 2012), pp. 33-47



Quantized vortices in: (a) a BEC of bosonic sodium atoms, a fermionic condensate of ^{6}Li atoms in the BEC (b) and BCS (c) states.

Zwierlein et al, Nature 435, 1047 (2005)

Summary

Superfluidity and superconductivity are intimately related **macroscopic quantum phenomena**:

- absence of electric resistance/viscosity,
- persistent current/flow in rings,
- Meissner-Ochsenfeld effect ($\boldsymbol{B} = 0$)/Hess-Fairbank effect ($\boldsymbol{L} = 0$),
- critical current/velocity,
- quantized magnetic flux tubes/vortex lines.

The dynamics of superfluids and superconductors (charged superfluids) can be explained by a **two-fluid model**.

Superfluidity and superconductivity are associated with **Bose-Einstein condensation**. In the case of fermions, the condensation proceeds through the formation of **Cooper pairs**.